MULTISCALE NUMERICAL METHODS FOR PARTIAL DIFFERENTIAL EQUATIONS USING LIMITED GLOBAL INFORMATION AND THEIR APPLICATIONS

A Dissertation

by

LIJIAN JIANG

Submitted to the Office of Graduate Studies of
Texas A&M University
in partial fulfillment of the requirements for the degree of

DOCTOR OF PHILOSOPHY

August 2008

Major Subject: Mathematics

MULTISCALE NUMERICAL METHODS FOR PARTIAL DIFFERENTIAL EQUATIONS USING LIMITED GLOBAL INFORMATION AND THEIR APPLICATIONS

A Dissertation

by

LIJIAN JIANG

Submitted to the Office of Graduate Studies of
Texas A&M University
in partial fulfillment of the requirements for the degree of

DOCTOR OF PHILOSOPHY

Approved by:

Chair of Committee,	Yalchin Efendiev
Committee Members,	Raytcho Lazarov
	Binayak Mohanty
	Jay Walton
Head of Department,	Al Boggess

August 2008

Major Subject: Mathematics

ABSTRACT

Multiscale Numerical Methods for Partial Differential Equations

Using Limited Global Information and Their Applications. (August 2008)

Lijian Jiang, B.S., Hunan Normal University;

M.S., Institute of Mathematics, Chinese Academy of Sciences

Chair of Advisory Committee: Dr. Yalchin Efendiev

In this dissertation we develop, analyze and implement effective numerical methods for multiscale phenomena arising from flows in heterogeneous porous media. The main purpose is to develop innovative numerical and analytical methods that can capture the effect of small scales on the large scales without resolving the small scale details on a coarse computational grid. This research activity is strongly motivated by many important practical applications arising in contaminant transport in heterogeneous porous media, oil reservoir simulations and subsurface characterization.

In the work, we investigate three main multiscale numerical methods, i.e., multiscale finite element method, partition of unity method and mixed multiscale finite element method. These methods employ limited single or multiple global information. We apply these numerical methods to partial differential equations (elliptic, parabolic and wave equations) with continuum scales. To compute the solution of partial differential equations on a coarse grid, we define global fields such that the solution smoothly depends on these fields. The global fields typically contain non-local information required for achieving a convergence independent of small scales. We present a rigorous analysis and show that the proposed global multiscale numerical methods converge independent of small scales. In particular, a global mixed multiscale finite element method is extensively studied and applied to two-phase flows.

We present some numerical results for two-phase simulations on coarse grids. The numerical results demonstrate that the global multiscale numerical methods achieve high accuracy.

ACKNOWLEDGMENTS

I would like to foremost thank Dr. Yalchin Efendiev, my supervisor, for his guidance, many suggestions and constant help throughout my Ph.D study. His support and insightful talking with me were crucial to the successful completion of the research. I am also thankful to Dr. Raytcho Lazarov and Dr. Jay Walton, not only for serving on my committee, but also for teaching me numerical methods and mathematical modeling. I also wish to thank Dr. Binayak Mohanty for serving on my committee and providing many insightful comments.

Dr. Craig C. Douglas taught me data assimilation methods and provided insightful suggestion for my research. Dr. Jean-Luc Guermond taught me mathematical theory of finite element methods and discontinuous Galerkin methods and provided many interesting discussions. Dr. Joe Pasciak taught me mixed finite element methods. I would like to thank them and all the professors who taught me at Texas A&M University. I also wish to thank Ms. Monique Stewart for her incredible patience and her help.

I would like to thank Dr. Thomas Yizhao Hou and Dr. Jorg Aarnes for their help in my research. I am also grateful to Professor Yuzan He, my master supervisor in Chinese Academy of Sciences, and Dr. Xinhan Dong, for their everlasting support and encouragement.

I would like to thank my fellow graduate students, Yan Li, Dukjin Nam, Dr. Victor Ginting and Dr. Paul Dostert, for their help and collaboration. I also wish to thank all my friends for helping me during my Ph.D studies.

I have to express my deep gratitude to my wife Yan Li and my parents for their love and patience. Without their support and encouragement my research might not have started and progressed as smoothly. I am particularly thankful to my wife Yan

Li for spending her valuable time on supporting my research.

TABLE OF CONTENTS

Page

ABSTRACT iii

ACKNOWLEDGMENTS v

TABLE OF CONTENTS vii

LIST OF TABLES x

LIST OF FIGURES xi

CHAPTER

I INTRODUCTION 1

II BACKGROUND MATERIALS 7

2.1. Flow Equations in Porous Media 7
2.2. Function Space Notations and Some Inequalities 10

III MULTISCALE NUMERICAL METHODS FOR ELLIPTIC EQUATIONS USING GLOBAL INFORMATION AND APPLICATIONS TO TWO-PHASE FLOWS 16

3.1. Motivation of MsFEM Using Limited Global Information 17
3.2. MsFEM Using Single Global Information 20
3.2.1. Galerkin MsFEM Using Single Global Information 20
3.2.2. Mixed MsFEM Using Single Global Information . . 29
3.3. Multiscale Methods Using Multiple Global Information for Elliptic Equations 44
3.3.1. PUM Using Multiple Global Information 44
3.3.2. Harmonic Coordinate System Methods 55
3.3.3. Mixed MsFEM Using Multiple Global Information 58
3.4. Numerical Results 75
3.4.1. Discretization of Saturation Equation 75
3.4.2. Numerical Results Using Single Global Information 77

CHAPTER Page

3.4.3. Numerical Results Using Multiple Global Information 85
3.5. Conclusions and Comments 90

IV MULTISCALE NUMERICAL METHODS FOR PARABOLIC EQUATIONS WITH CONTINUUM SCALES USING GLOBAL INFORMATION 92

4.1. Preliminaries on Parabolic Equations 93
4.2. Galerkin MsFEM with Limited Global Information 95
4.3. PUM for Parabolic Equations Using Multiple Global Information 101
4.3.1. Framework of PUM for Parabolic Equations 101
4.3.2. Incorporating Multiple Global Information 103
4.3.3. Space and Time Dependent Global Information . . 113
4.3.4. Time Discretization of PUM for Parabolic Equations 117
4.4. Mixed MsFEM for Parabolic Equations Using Global Information 123
4.5. Numerical Results 127
4.6. Conclusions and Comments 130

V MULTISCALE NUMERICAL METHODS FOR WAVE EQUATIONS WITH CONTINUUM SCALES USING GLOBAL INFORMATION 135

5.1. Preliminaries on Wave Equations 136
5.2. Galerkin MsFEM with Limited Global Information 138
5.3. PUM for Wave Equations Using Multiple Global Information 142
5.3.1. Framework of PUM for Wave Equations 143
5.3.2. Incorporating Multiple Global Information 144
5.3.3. Time Discretization of PUM for Wave Equations . 151
5.4. Mixed MsFEM for Wave Equations Using Multiple Global Information 154
5.5. Numerical Results 163
5.6. Conclusions and Comments 166

VI CONCLUSIONS 167

6.1. Conclusions 167
6.2. Future Work 168

REFERENCES 171

CHAPTER Page

VITA 178

LIST OF TABLES

TABLE Page

3.1 Relative Errors (layer=40, $\frac{\mu_w}{\mu_0} = 1/3$) 83

3.2 Relative Errors (layer=50, $\frac{\mu_w}{\mu_0} = 1/3$) 83

3.3 Relative Errors (layer=70, $\frac{\mu_w}{\mu_0} = 1/3$) 84

3.4 Relative Errors (layer=40, $\frac{\mu_w}{\mu_0} = 1/10$) 84

3.5 Relative Errors (layer=50, $\frac{\mu_w}{\mu_0} = 1/10$) 84

3.6 Relative Errors (layer=70, $\frac{\mu_w}{\mu_0} = 1/10$) 85

4.1 Relative Errors 130

5.1 Relative Errors at Time=0.4, Coarse Model = 22x10 165

5.2 Relative Errors at Time=0.6, Coarse Model = 55x15 165

LIST OF FIGURES

FIGURE Page

3.1 Schematic description of basis function 22

3.2 Schematic description of patch 52

3.3 Comparison of saturation between reference solution and MsFEM solution at $PVI = 1$, layer=50, 12×11 coarse grid and $\frac{\mu_w}{\mu_o} = 1/3$; Top: The reference saturation; Middle: The saturation using global mixed MsFEM; Bottom: Multiscale saturation using local mixed MsFEM. 79

3.4 Relative saturation error, layer=50, 12×11 coarse grid and $\frac{\mu_w}{\mu_o} = 1/3$ 80

3.5 Water-cut curve, layer=50, 12×11 coarse grid and $\frac{\mu_w}{\mu_o} = 1/3$ 80

3.6 Comparison of saturation between reference solution and MsFEM solution at $PVI = 1$, layer=50, 12×11 coarse grid and $\frac{\mu_w}{\mu_o} = 1/10$; Top: The reference saturation; Middle: The saturation using global mixed MsFEM; Bottom: Multiscale saturation using local mixed MsFEM. 81

3.7 Relative saturation error, layer=50, 12×11 coarse grid and $\frac{\mu_w}{\mu_o} = 1/10$ 82

3.8 Water-cut curve, layer=50, 12×11 coarse grid and $\frac{\mu_w}{\mu_o} = 1/10$ 82

3.9 Top: Comparison of water-cut between reference solution and multiscale solution; Middle: The reference saturation at $PVI = 1$; Bottom: Multiscale saturation at $PVI = 1$. (layer 85) 88

3.10 The saturation and water-cut error using one single-phase flow solution and two single-phase flow solutions, $\mu_o/\mu_w = 0.1$. (layer 85) 88

3.11 The saturation and water-cut error using one single-phase flow solution and two single-phase flow solutions, $\mu_o/\mu_w = 10$. (layer 85) . 89

FIGURE Page

3.12 The saturation and water-cut error using one single-phase flow solution and two single-phase flow solutions, $\mu_o/\mu_w = 10$ for different degree of coarsening. (layer 85) 90

4.1 Log of permeability field of layer 40 of SPE 10 SPE comparative project [24] 128

4.2 Pressure at $t = 0.12$. Left: Reference solution. Right: The solution obtained with MsFEM using limited global information. 131

4.3 Horizontal flux at $t = 0.12$. Left: Reference horizontal flux. Right: The horizontal flux obtained with MsFEM using limited global information. 131

4.4 Vertical flux at $t = 0.12$. Left: Reference vertical flux. Right: The vertical flux obtained with MsFEM using limited global information. 132

4.5 Pressure at $t = 0.32$. Left: Reference solution. Right: The solution obtained with MsFEM using limited global information. 132

4.6 Horizontal flux at $t = 0.32$. Left: Reference horizontal flux. Right: The horizontal flux obtained with MsFEM using limited global information. 133

4.7 Vertical flux at $t = 0.32$. Left: Reference vertical flux. Right: The vertical flux obtained with MsFEM using limited global information. 133

5.1 Solution u_h at $t = 0.4$. Left: Reference solution. Middle: solution obtained by global MsFEM. Right: solution by local MsFEM 165

5.2 Solution u_h at $t = 0.6$. Left: Reference solution. Middle: solution obtained by global MsFEM. Right: solution by local MsFEM 166

CHAPTER I

INTRODUCTION

Subsurface flows are often affected by heterogeneities in a wide range of length scales. It is therefore difficult to resolve numerically all of the scales that impact flow and transport. Typically, upscaled or multiscale models are employed for such systems. The main idea of upscaling techniques is to form coarse-scale equations with a prescribed analytical form that may differ from the underlying fine-scale equations. In multiscale methods, the fine-scale information is carried throughout the simulation and the coarse-scale equations are generally not expressed analytically, but rather formed and solved numerically. In the case of scale separation, one can localize the computation of effective parameters or basis functions. However, these approaches do not perform well if there is no scale separation, and some type of limited global information is needed for representing the distant/non-local effects. In this dissertation, we present multiscale numerical methods which incorporate limited global information stemming from the heterogeneities. We consider the following multiscale numerical methods: multiscale finite element method (MsFEM), mixed multiscale finite element method (mixed MsFEM) and partition of unity method (PUM).

A multiscale finite element method and a mixed multiscale finite element method were first introduced in [38] and [22] respectively. The main idea of multiscale finite element methods is to incorporate the small scale information into finite element basis functions and couple them through a global formulation of the problem. The multiscale method in [38] shares some similarities with a number of multiscale numerical methods, such as residual free bubbles [18, 56], variational multiscale method [40],

This dissertation follows the style of the Journal of Computational Physics.

two-scale finite element methods [48], two-scale conservative subgrid approaches [9], and multiscale mortar methods [11]. We remark that special basis functions in finite element methods have been used earlier in [15, 14]. The multiscale finite element methodology has been modified and successfully applied to two-phase flow simulations in [43, 42, 22, 1] and extended to nonlinear partial differential equations [33, 30]. Arbogast in [9] used a variational multiscale strategy to construct a multiscale method for two-phase flow simulations.

One of the fundamental issues in multiscale simulations is an accurate capturing of subgrid effects. It is known (e.g., [38]) that the local methods suffer from the resonance error. The resonance errors usually exhibit themselves as the ratio between the coarse mesh size and the characteristic length scale. If the mesh size is close to a characteristic length scale, multiscale methods that use only local information do not converge when the ratio between the mesh size and the characteristic length scale is kept fixed. To develop multiscale methods which converge without the resonance error, some type of limited global information is needed in the construction of basis functions. In a number of recent papers [1, 21, 29, 53, 59], limited global information has been successfully used for developing multiscale finite element methods that converge without the resonance error. These methods are applicable for problems without scale separation. However, in order to apply these methods to multi-phase flow simulations (described in Section 2.1), one needs to develop basis functions for the velocity field which are conservative and can be used throughout two-phase flow simulation without updating them. Velocity basis functions are used in solving the transport equations of the phases. For simulations of two-phase flow and transport, it is important to construct multiscale basis functions for the velocity field that are conservative.

In this dissertation, we present a framework for incorporating single and multiple global information which is represented by a set of functions. These functions can be thought as auxiliary functions which contain essential information about the heterogeneities. One can use local auxiliary functions in the case of scale separation, though our focus in this work is in the use of global information. We would like to note that the computations of the global fields are performed off-line. The basis functions constructed employing the global solutions can be used to solve flow equations with different source terms, boundary conditions or mobility (see Section 2.1 for definition) on the coarse grid. The latter is similar to upscaling techniques where the effective parameters are computed based on media properties and they can be used for different flow scenarios or in two-phase or multi-phase flow simulations. The computation of global fields requires solving single-phase equations, thus the proposed approaches are effective when flow equations are solved multiple times. One can use local solutions instead of global solutions in the proposed formulation of the global multiscale numerical methods. In this respect, the proposed global multiscale numerical methods will be similar to previously introduced multiscale methods in [38, 22] and the resonance error will appear in the convergence analysis.

In this dissertation, we consider elliptic, parabolic and wave equations with continuum spatial scales. Assuming that the solution of the partial differential equation smoothly depends on a single global field or a number of global fields, we construct basis functions which span these global fields. We discuss several cases where the global field/fields can be found and it can be shown that the solution smoothly depends on these global field/fields. Examples are shown for both deterministic and parameter dependent flow equations. We present rigorous analysis of the proposed multiscale numerical methods. We show that these methods are stable and converge without the resonance error. We present a few preliminary numerical results to demonstrate the efficiency of the proposed approaches. We also consider a parameter dependent permeability field (a simplified case for general stochastic permeability field) where a limited number of parameter value is used to generate the global fields. Based on these global fields, we compute multiscale basis functions. Basis functions are computed using single-phase flow solutions and these basis functions are used for the simulations of two-phase flow and transport. We consider SPE Comparative Project (also called SPE 10) [24]. These permeability fields are channelized and difficult to upscale and, therefore, single-phase flow information (limited global information) is used. Our numerical results show that one can achieve high accuracy with a few global fields corresponding to single-phase flow solutions.

The dissertation is organized as follows.

In Chapter II, we recall some preliminary background materials. We introduce flow equations in porous media, some function space notations and some inequalities that will be studied throughout the dissertation. First, we present the flow equations (e.g., two-phase immiscible flow) in porous media which have many applications in petroleum engineering and subsurface modeling. Second, we present some function space notations and inequalities, which are often utilized in the analysis of the dissertation.

In Chapter III, we present MsFEM and mixed MsFEM using single global information for two-phase flow equations. These methods use global information from an initial state to accurately model two-phase immiscible flow dynamics in heterogeneous porous media. We present analysis of both a Galerkin multiscale finite element method and a mixed multiscale finite element method. The analysis assumes that the fine-scale features of two-phase flow dynamics strongly depend on the initial state, or equivalently on the corresponding single-phase flow solution at initial time. The assumption is relaxed for the case with scale separation. We provide an extension of MsFEMs to the case where multiple global information is used. We consider Galerkin

MsFEM, PUM, harmonic coordinate system method and mixed MsFEM. We note that PUM using global information is not a trivial extension of MsFEM that employs limited global information because the construction of basis function in every "patch" are different from that global MsFEM discussed previously, i.e., basis functions in each "patch" are the span of multiplication of partial unity functions and global fields. We apply these multiscale methods to elliptic equations with continuum spatial scales. The global fields typically contain small scale (local or global) information required for achieving a convergence with respect to the coarse mesh size. We present a rigorous analysis for these global multiscale numerical methods and show that the proposed numerical methods converge. In particular we extensively explore the global mixed multiscale methods and its application to two-phase flows. Some preliminary numerical results for global mixed MsFEM are shown. As for spatial heterogeneities, channelized permeability fields (SPE 10) with strong non-local effects are considered.

In Chapter IV, we consider multiscale numerical approaches introduced in Chapter III for solving parabolic equations with heterogeneous coefficients. Our interest stems from porous media applications and we assume that there is no scale separation with respect to spatial variables. To compute the solution of these multiscale problems on a coarse grid, we define global fields such that the solution smoothly depends on these fields. We present various finite element discretization techniques and provide analysis of these methods. A few representative numerical examples are presented using heterogeneous fields with strong non-local features. These numerical results demonstrate that the solution can be captured more accurately on the coarse grid when some type of limited global information is used.

In Chapter V, we explore the proposed global multiscale approaches for solving wave equations with heterogeneous coefficients. Our interest comes from geophysics applications and we assume that there is no scale separation with respect to spatial variables. We present various multiscale finite element discretization techniques and provide analysis of these methods. A few representative numerical examples are presented using heterogeneous fields with strong non-local features.

In Chapter VI, we draw some conclusions for the dissertation and come up with some problems for further research.

CHAPTER II

BACKGROUND MATERIALS

In this chapter we introduce some background materials. Section 2.1 introduces flow and transport equations in porous media. Section 2.2 introduces function space notations and some inequalities.

2.1. Flow Equations in Porous Media

In this section, we present single-phase and two-phase flow equations neglecting the effects of gravity, compressibility, capillary pressure and dispersion on the fine scale. Porosity, defined as the volume fraction of the void space, will be taken to be constant and therefore serves only to rescale time. The two phases will be referred to as water and oil and designated by the subscripts w and o, respectively. We can then write Darcy's law, with all quantities dimensionless, for each phase j as follows:

$$u_j = -\lambda_j(S)k\nabla p, \tag{2.1}$$

where u_j $(j = o, w)$ is phase velocity, S is water saturation (volume fraction), p is pressure, $\lambda_j = k_{rj}(S)/\mu_j$ is phase mobility, where k_{rj} and μ_j are the relative permeability and viscosity of phase j respectively, and k is the permeability tensor.

Combining Darcy's law with conservation of mass, $div(u_w + u_o)$=0, allows us to write the flow equation in the following form

$$div(\lambda(S)k\nabla p) = f, \tag{2.2}$$

where the total mobility $\lambda(S)$ is given by $\lambda(S) = \lambda_w(S)+\lambda_o(S)$ and f is a source term. The saturation dynamics affects the flow equations. One can derive the equation describing the dynamics of the saturation

$$\frac{\partial S}{\partial t} + div(F) = 0, \tag{2.3}$$

where $F = uf_w(S)$, with $f_w(S)$, the fractional flow of water, given by $f_w = \lambda_w/(\lambda_w + \lambda_o)$, and the total velocity u by:

$$u = u_w + u_o = -\lambda(S)k\nabla p. \tag{2.4}$$

In the presence of capillary effects, an additional diffusion term is present in (2.3).

If $k_{rw} = S$, $k_{ro} = 1 - S$ and $\mu_w = \mu_o$, then the flow equation reduces to a single phase flow equation

$$div(k\nabla p^{sp}) = f.$$

This equation, the linear advection pollutant transport equation, will be referred to as the single-phase flow equation associated with (2.2), and p^{sp} will be referred to as the single-phase flow solution.

Remark 2.1.1. *The general governing equations for the single phase flow of a fluid in porous media include the conservation of mass, Darcy's law and an equation of state. Denote by ϕ the porosity of a porous medium, by ρ the density of the fluid per unit volume, by u the superficial Darcy velocity, and by f the external sources and sinks. Then the mass conservation equation of the porous medium can be formulated as*

$$\frac{\partial(\phi\rho)}{\partial t} = -div(\rho u) + f. \tag{2.5}$$

We can state the momentum conservation in Darcy's law, which indicates a linear relationship between the fluid velocity and the pressure head gradient. The Darcy's law is written as

$$u = -\frac{1}{\mu}k(\nabla p - \rho g\nabla z), \tag{2.6}$$

where k is the absolute permeability tensor of the porous medium, μ is the fluid viscosity, g is the magnitude of the gravitational acceleration, z is the depth. If $k = a(x)I$, where $a(x)$ is a scalar function and I is the identity matrix, then the medium is isotropic; otherwise, it is anisotropic. Let c_f denote the fluid compressibility. Then an equation of state is expressed by

$$\frac{1}{\rho}\frac{\partial \rho}{\partial p}|_T = c_f, \tag{2.7}$$

where T is a fixed temperature. (2.5), (2.6) and (2.7) give rise to a closed system.

Remark 2.1.2. *The general equations for two-phase flow include the following four equations.*

$$\left\{\begin{array}{rcl} \frac{\partial(\phi\rho_j S_j)}{\partial t} & = & -div(\rho_j u_j) + f_j \\ u_j & = & -\frac{1}{\mu_j}k_j(\nabla p_j - \rho_j g \nabla z) \\ p_c & = & p_o - p_w \\ S_w + S_o & = & = 1, \end{array}\right. \tag{2.8}$$

where $j = o, w$ and k_j is the effective permeability for phase j (w or o). Empirically, capillary pressure $p_c = p_c(S_w)$. The relationship among absolute permeability k, effective permeability k_j and relative permeability k_{rj} can be described by

$$k_j = k_{rj}k, \quad j = o, w.$$

The function k_{rj} shows the tendency of phase j to wet the porous medium.

As for the detailed description of flow and transport equations in porous media, we refer to [23].

2.2. Function Space Notations and Some Inequalities

In this section we review some function space notations and inequalities which will be frequently used throughout this thesis.

Let D be a domain. Denote by $L^p(D)$, $W^{k,p}(D)$, the usual Lebesgue space and Sobolev spaces. Let $W_0^{k,p}$ be the set of functions in $W^{k,p}(D)$ which vanish on ∂D, and $H^k(D) = W^{k,2}(D)$ and $H_0^k(D) = W_0^{k,2}(D)$. We denote $L^p(D)$-norm with $\|.\|_{0,p,D}$ and $\|.\|_{0,D} = \|.\|_{0,2,D}$ for $p = 2$. We denote $W^{k,p}$-norm with $\|.\|_{k,p,D}$ and $\|.\|_{k,D} = \|.\|_{k,2,D}$ for the norm of $H^k(D)$. Similarly, one can define the corresponding semi-norms by $|.|_{k,D}$ and $|.|_{k,p,D}$. We define the vector-valued Sobolev space by

$$\|f\|_{W^{m,p}(0,T;X)} := (\int_0^T \sum_{0\le k\le m} \|D_t^k f\|_X^p dt)^{\frac{1}{p}}$$

if X is a normed space and

$$|f|_{W^{m,p}(0,T;X)} := (\int_0^T \sum_{0\le k\le m} |D_t^k f|_X dt)^{\frac{1}{p}}$$

if X is a semi-norm space. If $p = 2$, we use $H^m(0,T;X)$ instead. When no ambiguity happens, we use the notation $W^{m,p}(X)$ to denote $W^{m,p}(0,T;X)$.

When we study the mixed finite element method, we often use the space $H(div, D)$, i.e.,

$$H(div, D) = \{u : \|u\|_{H(div,D)} := \|u\|_{0,D} + \|divu\|_{0,D} \le \infty\}.$$

If $\Gamma \subseteq \partial D$, we define $H_\Gamma^0(div, D) = \{u \in H(div, D) : u \cdot n_\Gamma = 0\}$. When no ambiguity happens, we set $H^0(div, D) = H_{\partial D}^0(div, D)$.

Lemma 2.2.1. (Poincaré Inequality)*[54] Let $1 \le p < \infty$ and D be a bounded domain in R^n. If $f \in W^{1,p}(D)$ and $f|_\Gamma = 0$, where $\Gamma \subset \partial D$ and measure of Γ, i.e.,*

$|\Gamma| > 0$, *then*

$$\|f\|_{0,p,D} \leq Cdiam(D)|f|_{1,p,D}, \tag{2.9}$$

where $C = C(n)$.

Remark 2.2.1. *Poincaré Inequality can be extended a slice domain* $L_d = \{x \in R^n : |x_n| \leq h\}$. *Let* $D \subset L_d$. *Then for* $f \in W_0^{1,p}(D)$, $1 \leq p \leq \infty$,

$$\|f\|_{0,p,D} \leq Ch|f|_{1,p,D},$$

where C *is independent of* n *and* p *[54].*

Remark 2.2.2. *Poincaré Inequality can be extended to multiplication of functions. Let* D *be bounded in* R^n. *If* $p, q \geq 2$ *such that* $\frac{1}{p} + \frac{1}{q} = 1$, *then for* $f \in W_0^{1,p}(D)$ *and* $g \in W_0^{1,q}(D)$,

$$\|fg\|_{0,1,D} \leq C_p diam(D)^p |f|^p_{1,p,D} + C_q diam(D)^q |g|^q_{1,q,D},$$

where $C_p = C(p, n)$, $C_q = C(q, n)$ *[54].*

Lemma 2.2.2. (Poincaré-Friedrichs Inequality)*[35] Let* D *be a bounded domain in* R^n. *Then for all* $f \in W^{1,p}(D)$, $1 \leq p < \infty$,

$$\|f - \langle f \rangle_D\|_{0,p,D} \leq Cdiam(D)|f|_{1,p,D}, \tag{2.10}$$

where $\langle f \rangle_D = \frac{1}{|D|}\int_D f dx$ *and* $C = C(n)$.

Remark 2.2.3. *Poincaré-Friedrichs Inequality can be generalized by using Petree-Tartar Lemma [34]. Let* L *be a continuous linear form on* $W^{1,p}(D)$ *whose restriction on constant (nonzero) functions is not zero. If* $L(1_D) = 1$, *where* $1_D \equiv 1$ *on* D. *Then for* $1 \leq p < \infty$,

$$\|f - L(f)\|_{1,p,D} \leq C|f|_{1,p,D},$$

where $C = C(D, p)$ *[34]. For example, we can take* $L(f) = \langle f \rangle_{D_0}$ *for any non-zero n-measure subset* $D_0 \subseteq D$ *or take* $L(f) = \langle f \rangle_{\partial D_1}$ *for any non-zero* $(n-1)$*-measure subboundary* $\partial D_1 \subseteq \partial D$.

Lemma 2.2.3. (Trace Inequality) *[6] Let* $\Gamma \subseteq \partial D$*. The trace operator* $\gamma : W^{k,p}(D) \longrightarrow W^{k-\frac{1}{p},p}(\Gamma)$ *is surjective and further*

$$\|f\|_{k-\frac{1}{p},p,\Gamma} \leq C\|f\|_{k,p,D}, \tag{2.11}$$

where $C = C(D, p, k)$.

Remark 2.2.4. *Let* D *be bounded in* R^n *and of class* C^k*. If* $f \in W^{k,p}(\Omega)$ $(kp < n)$*, then trace operator* $\gamma : W^{k,p}(D) \longrightarrow L_r(\partial D)$ *is surjective and*

$$\|f\|_{0,r,\partial D} \leq C\|f\|_{k,p,D},$$

where $r = \frac{(n-1)p}{n-kp}$ *and* $C = C(D, k, p)$.

Lemma 2.2.4. (Sobolev Embedding Theorem) *[6] Let* $D \subset R^n$ *be Lipschitz smooth and* Γ^k *be a k-dimensional domain obtained by intersecting* D *with a k-dimensional hyperplane in* R^n, $1 \leq k \leq n$*. Let* $j, m \in \{0\} \cup \{\mathbb{Z}^+\}$*. Then*

(1) If $0 < n - mp < k \leq n$ *and* $p \leq q \leq \frac{kp}{n-mp}$*, then*

$$W^{m+j,p}(D) \hookrightarrow W^{j,q}(\Gamma^k)$$

is continuously embedded. Furthermore if $1 \leq p < \infty$*, then*

$$W^{m+j,p}(D) \hookrightarrow W^{j,q}(\Gamma^k)$$

is compactly embedded.

(2) If $mp = n$, $1 \leq k \leq n$ *and* $p \leq q < \infty$*, then*

$$W^{m+j,p}(D) \hookrightarrow W^{j,q}(\Gamma^k)$$

is continuously embedded. Furthermore if $1 \leq p < \infty$*, then*

$$W^{m+j,p}(D) \hookrightarrow W^{j,q}(\Gamma^k)$$

is compactly embedded.

(3) If $mp > n$*, then*

$$W^{m+j,p}(D) \hookrightarrow \bar{C}^j(D) \equiv \{f \in C^j(D) : |D^\alpha f| < \infty, \quad \forall \quad |\alpha| \leq j\}$$

is continuously embedded. Furthermore if $1 \leq p < \infty$*, then*

$$W^{m+j,p}(D) \hookrightarrow \bar{C}^j(D)$$

is compactly embedded.

Remark 2.2.5. *Let* $\nu = [\frac{n}{p}] + 1 - \frac{n}{p}$ *if* $\frac{n}{p}$ *is not an integer and* ν *be any positive number less than 1 if* $\frac{n}{p}$ *is an integer. For* $kp > n$*, then*

$$W^{k,p}(D) \hookrightarrow C^{k-[\frac{n}{p}]-1,\nu}(\bar{D})$$

is continuously embedded, i.e.,

$$\|f\|_{C^{k-[\frac{n}{p}]-1,\nu}(\bar{D})} \leq C\|f\|_{k,p,D}, \tag{2.12}$$

where $C = C(D, k, p, n)$ *[35].*

We will also frequently use the following elementary inequalities.

Lemma 2.2.5. (Young's Inequality) *[54]. For* $a, b \geq 0$ *and* $\varepsilon > 0$ *and* $p, q > 1$ *with* $\frac{1}{p} + \frac{1}{q} = 1$*, then*

$$ab \leq \frac{1}{p}(\varepsilon a)^p + \frac{1}{q}(\frac{b}{\varepsilon})^q. \tag{2.13}$$

Lemma 2.2.6. (Jensen's Inequality) *[54] Let* $(D, \mathcal{A}, \mu)$ *be a probability measure space and* φ *be a real-valued convex function. If* f *is a real-valued* μ*-measurable* *function, then*

$$\varphi(\int_D f d\mu) \leq \int_D \varphi(f) d\mu. \tag{2.14}$$

Remark 2.2.6. *In numerical analysis, we often use the following version of Jensen's inequality: Let φ be a convex function and function $f : D \longrightarrow (0, \infty)$ and function $g : D \longrightarrow [0, \infty)$. Then*

$$\varphi(\frac{\int_D fg dx}{\int_D f dx}) \leq \frac{\int_D f\varphi(g) dx}{\int_D f dx}.$$

When we study evolution equation, we often use Gronwall's inequality.

Lemma 2.2.7. (Gronwall's Inequality) *[36] Let f, g and h be piecewise continuous nonnegative functions defined on an interval $a \leq t \leq b$, g being non-decreasing. If, for each $t \in [a, b]$,*

$$f(t) + h(t) \leq g(t) + \int_a^t f(s) ds,$$

then $f(t) + h(t) \leq e^{t-a} g(t)$.

Remark 2.2.7. *When we study discretization of evolution equation, we also need to use discrete analogue of Gronwall's Inequality:*

Let f, g and h be piecewise continuous nonnegative functions defined on

$$T_\Delta = \{t \in [0, T], \quad t = j\Delta t, \quad j = 0, 1, \cdots, m, \quad m\Delta t = T\},$$

g being non-decreasing. If

$$f(t) + h(t) \leq g(t) + C\Delta t \sum_{s=0}^{t-\Delta t} f(s),$$

where C is a positive constant. Then $f(t) + h(t) \leq e^{Ct} g(t)$ [36].

Throughout the dissertation, if we do not make explicit explanation, we will use Einstein summation convention, i.e., repeated indices are implicitly summed over.

This simplifies and shortens equations. For example,

$$a_i + b_i = \sum_i (a_i + b_i), \quad a_i b_i = \sum_i (a_i b_i)$$

and

$$a_{ik} b_{ij} = \sum_i (a_{ik} b_{ij}).$$

CHAPTER III

MULTISCALE NUMERICAL METHODS FOR ELLIPTIC EQUATIONS USING GLOBAL INFORMATION AND APPLICATIONS TO TWO-PHASE FLOWS

In this chapter, we propose some multiscale numerical methods and apply them to elliptic equations. These multiscale numerical methods employ limited global information. We present some multiscale numerical methods using one global or multiple global fields. The main results in the chapter are that the proposed multiscale numerical methods converge without assuming the scale separation and the convergence rate does not contain resonance errors. As an example in two-phase flow simulations, these methods use global information from an initial single-phase flow to accurately model two-phase immiscible flow dynamics in heterogeneous porous media. The analysis assumes that the fine-scale features of two-phase flow dynamics strongly depend on the initial state, or equivalently on the corresponding single-phase flow solution at initial time. The assumption is relaxed for the case with scale separation. These results can been found in our papers [4, 5, 3].

The chapter is organized as follows.

In Section 3.1, we provide the motivation of multiscale finite element methods using global information.

In Section 3.2, we introduce Galerkin MsFEM and mixed MsFEM using a single phase flow information. These multiscale techniques have advantages if the fine-scale features of two-phase flow dynamics strongly depend on the initial single-phase flow. In particular, we assume that two-phase flow pressure solution strongly depends on the initial single-phase pressure. The validity of this assumption is shown for channelized permeability in [29]. If there is a scale separation, then it is sufficient to assume that the direction of the average of the coarse-scale pressure gradient is unchanged during the course of a two-phase flow simulation (i.e., the coarse-scale streamlines does not change significantly during a simulation).

In Section 3.3, we develop partition of unity method using global information, numerical harmonic coordinate method and mixed MsFEM using multiple global information. These multiscale methods using multiple global information are not trivial extensions of the multiscale methods introduced in Section 3.2 where single global information is used because the construction of basis functions are different and the multiscale methods using multiple global fields have wider applications.

In Section 3.4, we use the global multiscale methods for simulation of two-phase flows and present the numerical results. We show that MsFEMs using global information are more accurate compared to MsFEMs which only use local information.

3.1. Motivation of MsFEM Using Limited Global Information

We study MsFEMs for elliptic equation

$$-div(\lambda(x)k(x)\nabla p) = f, \tag{3.1}$$

where $k(x)$ is a heterogeneous field and $\lambda(x)$ is assumed to be a smooth field. This equation is derived from two-phase flow equations when gravity and capillary effects are neglected (see (2.2)). Here p denotes the pressure. Our goal is to construct multiscale basis functions on the coarse grid (with grid size larger than the characteristic length scale of the problem) such that these basis functions can be used for various source terms $f(x)$, boundary conditions and mobilities $\lambda(x)$. For this reason, one typically looks for functions (local or global) which contain the essential information about the heterogeneities. For problems without scale separation, these functions are often the solutions of global problems, and thus, these methods are effective when

(3.1) is solved multiple times. For problems with scale separation, one can use the solutions of the local problem in constructing multiscale basis functions. The underlying assumption for these global fields used in the paper is the following. There exists N global fields p_1,..., p_N, such that

$$|p - G(p_1, ..., p_N)|_{1,\Omega} \leq C\delta, \tag{3.2}$$

where δ is sufficiently small, G is sufficiently smooth function, and p_1,.., p_N are solutions of $div(k(x)\nabla p) = 0$ with some prescribed boundary conditions. We note that δ refers to a physical parameters that shows how well the solution can be represented using some functions p_1,..., p_N. For example, in homogenization setting, δ is related to the smallest scales representing the heterogeneities. In some cases, the solution can be represented exactly with the global fields (see below) and in this case $\delta = 0$. Also, we note that the assumption (3.2) can be formulated for each coarse patch, where p_1, ..., p_N are different for each patch. In this case, p_1, ..., p_N are defined on coarse patches and the resulting multiscale method (introduced later) is similar to multiscale methods introduced earlier in [22]. Next, we briefly discuss the assumption (3.2).

In [29], it was shown that for channelized permeability fields, p is a smooth function of single-phase flow pressure (i.e., $N = 1$), where single-phase pressure equation is described by $div(k(x)\nabla p) = 0$ with boundary conditions as those corresponding to two-phase flow. In a general setting, it was shown by Owhadi and Zhang [53] that for an arbitrary smooth $\lambda(x)$, the solution is a smooth function of d linearly independent solutions of single-phase flow equations ($N = d$), where d is the space dimension. When considering random permeability fields, the permeability field is typically parameterized with a parameter that represents the uncertainties. In this case, we deal with a family of rough permeability fields such as $k = k(x, \theta)$, where θ

is in high dimensional space. For example, log-Gaussian permeability fields can be characterized using Karhunen-Loéve expansion [31] as

$$k(x, \theta_1, ..., \theta_M) = \exp(\theta_i \phi_i(x)),$$

where $\phi_i(x)$ are pre-computed spatial fields which depend on covariance matrix. In many of these parameterized cases, $k(x, \theta)$ is a smooth functions of $\theta = (\theta_1, ..., \theta_M)$, and thus one can use sparse approximation techniques to represent the solution on the coarse grid via the basis functions computed for a selected number of single-phase flow solutions (see [20, 27]).

If there is scale separation, one can use homogenization theory. In this case, p_i are given by the solutions of the local problems in a period (cf. [8]). More precisely, if $k_\epsilon(x) = k(x/\epsilon)$, where $k(y)$ is a periodic function in a unit cube Y, then p_i are solutions of

$$div(k(x)\nabla p_i) = 0, \;\; i = 1, ..., d,$$

such that $p_i = x_i + P_i$ with P_i is being a periodic function. In general, one can also use the solutions of local problems in coarse grid blocks in the construction of basis functions. In this case, the proposed method is similar to mixed MsFEM proposed in [22].

In the above assumption (3.2), p_i are solutions of flow equations. We denote the corresponding velocity field by u_i, i.e., $u_i = k\nabla p_i$. Then, the above assumption can be written in the following way. There exist sufficiently smooth scalar functions $A_1(x)$, ..., $A_N(x)$, such that the velocity corresponding to (3.1) can be written as

$$\|u - A_1(x)u_1 - ... - A_N(x)u_N\|_{0,\Omega} \leq C\delta. \tag{3.3}$$

We note that in the case of homogenization problems $u_i = k(x)\nabla p_i$, and one can easily show that $A_i(x) = \lambda(x)\frac{\partial p_0}{\partial x_i}$ and $\delta = \sqrt{\epsilon}$ ([46]). Here p_0 is the solution of the homogenized equation and thus A_i are smooth functions. In general case, $A_i(x) = \lambda(x)\frac{\partial G}{\partial p_i}$. In the analysis, we will use a stronger version of (3.3).

3.2. MsFEM Using Single Global Information

In this section, we discuss Galerkin MsFEM and mixed MsFEM using single-phase flow information and their applications to two-phase immiscible flow.

3.2.1. Galerkin MsFEM Using Single Global Information

Construction of MsFEM Basis Functions . The key idea of Galerkin MsFEM is the construction of basis functions on coarse grids that capture small-scale information. The basis functions are constructed from the solution of the leading order homogeneous elliptic equation on each coarse element with some specified boundary conditions. For further analysis, K denotes a generic coarse element and τ_h is a quasi-uniform family of coarse elements. Thus, if we consider a coarse element K that has d vertices x_j, the local basis functions $\phi_i, i = 1, \cdots, d$ are set to satisfy the following elliptic problem

$$\begin{cases} div(k(x)\nabla\phi_i^K) & = & 0 \quad \text{in} \quad K \\ \phi_i^K & = & d_i^K \quad \text{on} \quad \partial K, \\ \phi_i^K(x_j) & = & \delta_{ij}, \end{cases} \tag{3.4}$$

where $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ if $i \neq j$.

The function d_i^K for each i has been defined in various ways. e.g., in [43] it is chosen to vary linearly along ∂K, or to be the solution of a local one-dimensional problems. A solution of the problem in a slightly larger domain has also been used to define boundary conditions [38]. It can be shown that if d_i^K varies linearly along

∂K, then the multiscale finite element method has a resonance error [38].

We would like to note that an approximate solution of (3.4) can be used. For example, in the case of periodic or scale separation cases, the basis functions can be approximated using homogenization expansion [29]. This type of simplification is not applicable for problems considered in this thesis.

Next, we briefly describe multiscale finite element method using information from a single-phase flow solution. For simplicity, we restrict ourselves to the two dimensional case. For this method, d_i^K is the linear interpolation of p^{sp} using the values of $p^{sp}(x_j)$ $(j = 1, \cdots, d)$. In particular, for each element K (see Figure 3.1) with vertices x_i $(i = 1, ..., d)$ denote by $\phi_i^K(x)$ a restriction of the nodal basis on K, such that $\phi_i^K(x_j) = \delta_{ij}$. At the edges where $\phi_i^K(x) = 0$ at both vertices, we take boundary condition for $\phi_i^K(x)$ to be zero. Consequently, the basis functions are localized. We only need to determine the boundary condition at two edges that have the common vertex x_i $(\phi_i^K(x_i) = 1)$. Denote these two edges by $[x_{i-1}, x_i]$ and $[x_i, x_{i+1}]$. We only need to describe the boundary condition, d_i^K, for the basis function ϕ_i^K, along the edges $[x_i, x_{i+1}]$ and $[x_i, x_{i-1}]$. If $p^{sp}(x_i) \neq p^{sp}(x_{i+1})$, then

$$d_i^K(x)|_{[x_i,x_{i+1}]} = \frac{p^{sp}(x) - p^{sp}(x_{i+1})}{p^{sp}(x_i) - p^{sp}(x_{i+1})}, \quad d_i^K(x)|_{[x_i,x_{i-1}]} = \frac{p^{sp}(x) - p^{sp}(x_{i-1})}{p^{sp}(x_i) - p^{sp}(x_{i-1})}.$$

The case $p^{sp}(x_i) = p^{sp}(x_{i+1}) \neq 0$ and others can also be described (see [29]).

We define the Galerkin finite element space by

$$V_h = span\{\phi_i^K : 1, \cdots, d; K \in \tau_h\}. \tag{3.5}$$

The weak formulation of (2.2) is to seek $p_h \in V_h$ such that

$$(\lambda k^\epsilon \nabla p_h, \nabla q_h) = (f, q_h) \qquad \text{for any } q_h \in V_h, \tag{3.6}$$

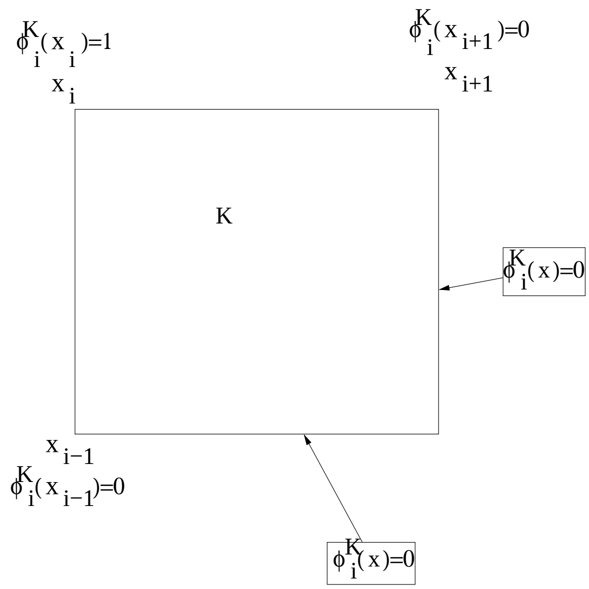

Fig. 3.1. Schematic description of basis function

where $(\cdot, \cdot)$ denotes inner product in L^2. In this case, we have the Cea's estimate [25]

$$|p - p_h|_{1,\Omega} \leq C \inf_{q_h \in V_h} |p - q_h|_{1,\Omega}, \tag{3.7}$$

where p is the solution of two-phase flow (2.2) and p_h is the numerical solution defined in (3.6). Throughout C denotes a generic constant independent of mesh size. Using (3.7), one can show $p_h^{sp} = p^{sp}$ in each coarse block K (see [29]) for the case with zero source term and non-zero boundary conditions. If the source term is not zero and in $L^2(\Omega)$, it can be easily shown that $|p_h^{sp} - p^{sp}|_{1,\Omega} \leq Ch\|f\|_{0,\Omega}$.

Convergence Analysis of MsFEM for Continuum Scale . We discuss the convergence for the case of continuum scale. We will use the following assumption for the case of continuum scale.

Assumption G. There exists a sufficiently smooth scalar valued function $G(\eta)$ *(*$G \in W^{3,\frac{2s}{s-2}}$*,* $s > 2$ *), such that*

$$|p - G(p^{sp})|_{1,\Omega} \leq C\delta, \tag{3.8}$$

where δ is sufficiently small.

Note that G is defined on a bounded interval because p^{sp} is a bounded function.

Following standard practice of finite element estimation, we seek $q_h = c_i\phi_i^K$, where ϕ_i^K are single-phase flow based multiscale finite element basis functions. Then from (3.7), we have

$$|p - p_h|_{1,\Omega} \le |p - G(p^{sp})|_{1,\Omega} + |G(p^{sp}) - c_i\phi_i^K|_{1,\Omega}. \tag{3.9}$$

Next, we present an estimate for the second term. We choose $c_i = G(p^{sp}(x_i))$, where x_i are vertices of K. Furthermore, using Taylor expansion of G around $\overline{p}_K$, which is the average of p^{sp} over K,

$$\begin{aligned} G(p^{sp}(x_i)) = G(\overline{p}_K) &+ G'(\overline{p}_K)(p^{sp}(x_i) - \overline{p}_K) \\ &+ (p^{sp}(x_i) - \overline{p}_K)^2 \int_0^1 sD''(p^{sp}(x_i) + s(\overline{p}_K - p^{sp}(x_i)))ds. \end{aligned} \tag{3.10}$$

We have in each K

$$\begin{aligned} c_i\phi_i^K &= G(\overline{p}_K)\sum_i \phi_i^K + G'(\overline{p}_K)(p^{sp}(x_i) - \overline{p}_K)\phi_i^K \\ &+ (p^{sp}(x_i) - \overline{p}_K)^2\phi_i^K \int_0^1 sD''(p^{sp}(x_i) + s(\overline{p}_K - p^{sp}(x_i)))ds \\ &= G(\overline{p}_K) + G'(\overline{p}_K)(p^{sp}(x_i)\phi_i^K - \overline{p}_K) \\ &+ (p^{sp}(x_i) - \overline{p}_K)^2\phi_i^K \int_0^1 sD''(p^{sp}(x_i) + s(\overline{p}_K - p^{sp}(x_i)))ds. \end{aligned} \tag{3.11}$$

In the last step, we have used $\sum_i \phi_i^K = 1$. Similarly, in each K,

$$\begin{aligned} G(p^{sp}(x)) = G(\overline{p}_K) &+ G'(\overline{p}_K)(p^{sp}(x) - \overline{p}_K) \\ &+ (p^{sp}(x) - \overline{p}_K)^2 \int_0^1 sD''(p^{sp}(x) + s(\overline{p}_K - p^{sp}(x)))ds. \end{aligned} \tag{3.12}$$

Using (4.11) and (4.12), we get

$$
\begin{aligned}
&|G(p^{sp}) - c_i\phi_i^K|_{1,K} \\
&\le |G'(\overline{p}_K)(p^{sp}(x) - p^{sp}(x_i)\phi_i^K)|_{1,K} \\
&+ |(p^{sp}(x_i) - \overline{p}_K)^2\phi_i^K \int_0^1 sD''(p^{sp}(x_i) + s(\overline{p}_K - p^{sp}(x_i)))ds|_{1,K} \\
&+ |(p^{sp}(x) - \overline{p}_K)^2 \int_0^1 sD''(p^{sp}(x) + s(\overline{p}_K - p^{sp}(x)))ds|_{1,K}.
\end{aligned}
\tag{3.13}
$$

Since $|p^{sp}(x) - p^{sp}(x_i)\phi_i^K|_{1,K} \le Ch\|f\|_{0,K}$, the estimate of the first term is the following

$$|G'(\overline{p}_K)(p^{sp}(x) - p^{sp}(x_i)\phi_i^K)|_{1,K} \le Ch\|f\|_{0,K}.$$

For the second term on the right hand side of (3.13), assuming $p^{sp}(x) \in W^{1,s}(\Omega)$, we have

$$
\begin{aligned}
&|(p^{sp}(x_i) - \overline{p}_K)^2\phi_i^K \int_0^1 sD''(p^{sp}(x_i) + s(\overline{p}_K - p^{sp}(x_i)))ds|_{1,K} \\
&\le Ch^{2-4/s}|p^{sp}|^2_{W^{1,s}(K)}|\phi_i^K|_{1,K} \\
&\le Ch^{1-2/s}|p^{sp}|_{W^{1,s}(\Omega)}|p^{sp}|_{W^{1,s}(K)} \\
&\le Ch^{1-2/s}|p^{sp}|_{W^{1,s}(K)}.
\end{aligned}
\tag{3.14}
$$

where $s > 2$ and the fact that $\phi_i^K \le C\frac{1}{h}$. Here, we have used the inequality (2.12)

$$|u(x) - u(y)| \le C|x - y|^{1-2/s}|u|_{W^{1,s}},$$

for $s > 2$, where C depends only on s.

For the third term on the right hand side of (3.13), A straightforward calculation gives

$$
\begin{aligned}
&|(p^{sp}(x)-\overline{p}_K)^2\int_0^1 sD''(p^{sp}(x)+s(\overline{p}_K-p^{sp}(x)))ds|_{1,K}\\
&\le \|(p^{sp}(x)-\overline{p}_K)^2\nabla p^{sp}(x)\int_0^1(1-s)sD'''(p^{sp}(x)+s(\overline{p}_K-p^{sp}(x)))ds\|_{0,K}\\
&+\|2(p^{sp}(x)-\overline{p}_K)\nabla p^{sp}(x)\int_0^1 sD''(p^{sp}(x)+s(\overline{p}_K-p^{sp}(x)))ds\|_{0,K}\\
&\le Ch^{2-\frac{4}{s}}|p^{sp}|_{1,s,K}(|p^{sp}|^s_{1,s,\Omega}+|G|^{\frac{2s}{s-2}}_{3,\frac{2s}{s-2}})^{\frac{1}{2}} \qquad (3.15)\\
&+Ch^{1-\frac{2}{s}}|p^{sp}|_{1,s,K}(|p^{sp}|^s_{1,s,\Omega}+|G|^{\frac{2s}{s-2}}_{3,\frac{2s}{s-2}})^{\frac{1}{2}}\\
&\le Ch^{2-\frac{4}{s}}|p^{sp}|_{1,s,K}+Ch^{1-\frac{2}{s}}|p^{sp}|_{1,s,K}\\
&\le Ch^{1-\frac{2}{s}}|p^{sp}|_{1,s,K},
\end{aligned}
$$

where we used Young's inequality (2.13) in the second step.

Combining the above estimates, we have for (3.13),

$$|G(p^{sp})-c_i\phi_i^K|_{1,K}\le Ch^{1-2/s}|p^{sp}|_{W^{1,s}(K)}+Ch\|f\|_{0,K}. \qquad (3.16)$$

Summing (3.16) over all K and taking into account Assumption G, we have

$$|p-p_h|_{1,\Omega}\le C\delta+Ch^{1-2/s}|p^{sp}|_{W^{1,s}(\Omega)}+Ch\|f\|_{0,\Omega}\le C\delta+Ch^{1-2/s}. \qquad (3.17)$$

Consequently, if $s>2$, single-phase flow based multiscale finite element method converges and we have the following theorem.

Theorem 3.2.1. *Under Assumption G and $p^{sp}\in W^{1,s}(\Omega)$ ($s>2$), multiscale finite element method converges with the rate given by (3.17).*

Remark 3.2.1. *We can relax the assumption on G. In particular, it is sufficient to assume $G\in W^{2,m}$ ($m\ge 1$). In this case, the proof can be carried out using Taylor polynomials in Sobolev spaces and will be given in later chapter. Also, if we assume $\nabla p^{sp}\in L^\infty(\Omega)$, then the convergence rate in (3.17) is $C\delta+Ch$.*

Convergence Analysis of MsFEM for Scale Separation Case . Now we discuss the scale separation case. For the analysis of the case with scale separation, it is sufficient to assume *Assumption G* for homogenized part of the pressure. Next, we briefly review homogenization. We consider a two-phase flow model problem in a scale-separation form

$$\begin{aligned} -div(\lambda(x)k^{\epsilon}(x)\nabla p_{\epsilon}) = f(x) \quad \text{in} \quad \Omega \\ p_{\epsilon} = 0 \quad \text{on} \quad \partial\Omega, \end{aligned} \tag{3.18}$$

where $\lambda(x)$ is a positive smooth function and $k^{\epsilon}(x) = k(x/\epsilon)$ is a symmetric, positive definite matrix and $k(y)$ is a periodic function, $y = x/\epsilon$. The corresponding single phase flow problem is

$$\begin{aligned} -div(k^{\epsilon}(x)\nabla p_{\epsilon}^{sp}) = f(x) \quad \text{in} \quad \Omega \\ p_{\epsilon}^{sp} = 0 \quad \text{on} \quad \partial\Omega. \end{aligned} \tag{3.19}$$

Using a formal multiscale expansion $p_{\epsilon}(x) = p_0(x) + \epsilon p_1(x, \frac{x}{\epsilon}) + \cdots$, one can obtain the homogenized equation associated with (3.18)

$$\begin{aligned} -div(\lambda(x)k^{*}\nabla p_0) = f(x) \quad \text{in} \quad \Omega \\ p_0 = 0 \quad \text{on} \quad \partial\Omega, \end{aligned} \tag{3.20}$$

where matrix $k^{*} = \{k^{*}_{ij}\}$ and $k^{*}_{ij} = \langle k_{ij} + k_{im}\frac{\partial \chi_j}{\partial y_m}\rangle_Y$, in which χ_j is Y-periodic and solves the auxiliary equation

$$div(k(y)\nabla\chi_j) = -div(k(y)e_j) \quad \text{in} \quad Y, \tag{3.21}$$

where e_j is the unit vector with 1 in j-th component. Similarly, one can derive the homogenized equation associated with (3.19),

$$-div(k^*\nabla p_0^{sp}) = f(x) \quad \text{in} \quad \Omega$$
$$p_0^{sp} = 0 \quad \text{on} \quad \partial\Omega. \tag{3.22}$$

Let us recall the standard MsFEM [38], for which d_i^K is chosen to vary linearly along ∂K authors in [39] obtained the following theorem.

Theorem 3.2.2. *Let p_ϵ and p_h solve the equation (3.18) and (3.6) respectively, then for $h > \epsilon$,*

(1) $\|p_\epsilon - p_h\|_1 \leq Ch\|f\|_0 + C\sqrt{\frac{\epsilon}{h}}$;

(2) $\|p_\epsilon - p_h\|_0 \leq Ch^2\|f\|_0 + C\epsilon + C\frac{\epsilon}{h}$.

Employing over-sampling technique to reduce resonance error, authors in [32] obtained that

$$\|p_\epsilon - p_h\|_{1,h} \leq C(h + \sqrt{\epsilon} + \frac{\epsilon}{h}).$$

Let us come back the MsFEM using limited global information for the scale separation problem. The *Assumption G*, for the case with scale separation, can be replaced by

Assumption G-S. There exists a sufficiently smooth scalar valued function $G(\eta)$ ($G \in W^{3,\frac{2s}{s-2}}$, $s > 2$) such that

$$|p_0 - G(p_0^{sp})|_{1,\Omega} \leq C\delta_0, \tag{3.23}$$

where δ_0 is sufficiently small.

The convergence estimate can be easily derived repeating the steps of no-scale separation case, if we estimate $|p_\epsilon - G(p_\epsilon^{sp})|_{1,\Omega}$.

Let $\chi = \{\chi_1, \cdots, \chi_n\}$. We introduce

$$\tilde{p}_\epsilon = p_0 + \epsilon\chi\nabla p_0, \quad \tilde{p}_\epsilon^{sp} = p_0^{sp} + \epsilon\chi\nabla p_0^{sp}.$$

Next, we write

$$|p_\epsilon - G(p_\epsilon^{sp})|_{1,\Omega} \le |p_\epsilon - \tilde{p}_\epsilon|_{1,\Omega} + |\tilde{p}_\epsilon - G(\tilde{p}_\epsilon^{sp})|_{1,\Omega} + |G(\tilde{p}_\epsilon^{sp}) - G(p_\epsilon^{sp})|_{1,\Omega}. \tag{3.24}$$

The first term on the right hand side of (3.24) can be estimated using (e.g., [46])

$$|p_\epsilon - \tilde{p}_\epsilon|_{1,\Omega} \le C\sqrt{\epsilon}|p_0|_{2,\Omega}. \tag{3.25}$$

For the estimation of the second term on the right hand side of (3.24), we have

$$\begin{aligned}
&|\tilde{p}_\epsilon - G(\tilde{p}_\epsilon^{sp})|_{1,\Omega} \le \|(I + \nabla_y \chi)\nabla p_0 - G'(\tilde{p}_\epsilon^{sp})(I + \nabla_y \chi)\nabla p_0^{sp}\|_{0,\Omega} \\
&+ \|\epsilon\chi\nabla^2 p_0\|_{0,\Omega} + \|G'(\tilde{p}_\epsilon^{sp})\epsilon\chi\nabla^2 p_0^{sp}\|_{0,\Omega} \\
&\le \|(I + \nabla_y \chi)(\nabla p_0 - G'(\tilde{p}_\epsilon^{sp})\nabla p_0^{sp})\|_{0,\Omega} + C\epsilon|p_0|_{2,\Omega} + C\epsilon|p_0^{sp}|_{2,\Omega} \\
&\le C\|\nabla p_0 - G'(\tilde{p}_\epsilon^{sp})\nabla p_0^{sp}\|_{0,\Omega} + C\epsilon|p_0|_{2,\Omega} + C\epsilon|p_0^{sp}|_{2,\Omega} \\
&\le C\|\nabla p_0 - G'(p_0^{sp})\nabla p_0^{sp}\|_{0,\Omega} + C\|(G'(p_0^{sp}) - G'(\tilde{p}_\epsilon^{sp}))\nabla p_0^{sp})\|_{0,\Omega} + C\epsilon|p_0|_{2,\Omega} + C\epsilon|p_0^{sp}|_{2,\Omega} \\
&\le C\|\nabla p_0 - G'(p_0^{sp})\nabla p_0^{sp}\|_{0,\Omega} + C\|p_0^{sp} - \tilde{p}_\epsilon^{sp}\|_{L^\infty(\Omega)}\|\nabla p_0^{sp}\|_{0,\Omega} + C\epsilon|p_0|_{2,\Omega} + C\epsilon|p_0^{sp}|_{2,\Omega} \\
&\le C\|\nabla p_0 - G'(p_0^{sp})\nabla p_0^{sp}\|_{0,\Omega} + C\epsilon\|\nabla p_0^{sp}\|_{0,\Omega} + C\epsilon|p_0|_{2,\Omega} + C\epsilon|p_0^{sp}|_{2,\Omega}.
\end{aligned} \tag{3.26}$$

For the third term on the right hand side of (3.24), we have

$$\begin{aligned}
|G(\tilde{p}_\epsilon^{sp}) - G(p_\epsilon^{sp})|_{1,\Omega} &\le \|(G'(p_\epsilon^{sp}) - G'(\tilde{p}_\epsilon^{sp}))\nabla p_\epsilon^{sp}\|_{0,\Omega} + \|G'(\tilde{p}_\epsilon^{sp})(\nabla p_\epsilon^{sp} - \nabla \tilde{p}_\epsilon^{sp})\|_{0,\Omega} \\
&\le C\|p_\epsilon^{sp} - \tilde{p}_\epsilon^{sp}\|_{\infty,\Omega}|\nabla p_\epsilon^{sp}|_{1,\Omega} + C|p_\epsilon^{sp} - \tilde{p}_\epsilon^{sp}|_{1,\Omega} \le C\epsilon|p_\epsilon^{sp}|_{1,\Omega} + C\sqrt{\epsilon}|p_0^{sp}|_{2,\Omega}.
\end{aligned} \tag{3.27}$$

Combining the above estimates, we have

$$\begin{aligned}
|p_\epsilon - G(p_\epsilon^{sp})|_{1,\Omega} \le C\sqrt{\epsilon}|p_0|_{2,\Omega} + \|\nabla p_0 - G'(p_0^{sp})\nabla p_0^{sp}\|_{0,\Omega} + \\
C\epsilon(|p_\epsilon^{sp}|_{1,\Omega} + |p_0^{sp}|_{1,\Omega}) + C\sqrt{\epsilon}|p_0^{sp}|_{2,\Omega}.
\end{aligned} \tag{3.28}$$

Following the error estimate derived for no-scale separation case we can obtain the estimate for $|G(p_\epsilon^{sp}) - c_i\phi_i^K|$. Thus, using Assumption G-S, we have

$$\begin{aligned}|p_\epsilon - c_i\phi_i^K|_{1,\Omega} \leq C\sqrt{\epsilon}|p_0|_{2,\Omega} + C\delta_0 + C\epsilon(|p_\epsilon^{sp}|_{1,\Omega} + |p_0^{sp}|_{1,\Omega}) + \\ C\sqrt{\epsilon}|p_0^{sp}|_{2,\Omega} + Ch^{1-2/s} \leq C\delta_0 + C\sqrt{\epsilon} + Ch^{1-2/s}.\end{aligned} \tag{3.29}$$

Thus, we have the following theorem.

Theorem 3.2.3. *Under Assumption G-S and the fact that $p_\epsilon^{sp} \in W^{1,s}(\Omega)$ $(s > 2)$ and $|p_0|_{2,\Omega} + |p_0^{sp}|_{2,\Omega}$ is bounded, multiscale finite element method converges with the rate given by (3.29).*

3.2.2. Mixed MsFEM Using Single Global Information

Mixed FEM . For simplicity, we assume Neumann boundary conditions. First, we review the mixed multiscale finite element formulation following [22] (see also [1, 10, 9]). We can rewrite two-phase flow equation as

$$\begin{cases} (\lambda k)^{-1}u - \nabla p &= 0 \quad \text{in} \quad \Omega \\ div(u) &= 0 \quad \text{in} \quad \Omega \\ \lambda(x)k(x)\nabla p \cdot n &= g(x) \quad \text{on} \quad \partial\Omega, \end{cases} \tag{3.30}$$

where $\lambda(x) \geq c > 0$ for some constant c and k is uniformly positive , symmetric and bounded. The variational problem associated with (3.30) is to seek $(u, p) \in H(div, \Omega) \times L^2(\Omega)/R$ such that $u \cdot n = g$ on $\partial\Omega$ and

$$\begin{aligned} ((\lambda k)^{-1}u, v) + (divv, p) = &\quad 0 \quad \forall v \in H_0(div, \Omega) \\ (divu, q) = &\quad 0 \quad \forall q \in L^2(\Omega)/R. \end{aligned} \tag{3.31}$$

By defining

$$a(u, v) = ((\lambda k)^{-1}u, v), \quad b(v, q) = (divv, q), \tag{3.32}$$

we can rewrite the weak formulation as

$$\begin{aligned} a(u,v) + b(v,p) &= 0 \quad \forall v \in H_0(div,\Omega), \\ b(u,q) &= 0 \quad \forall q \in L^2(\Omega)/R. \end{aligned}$$

Let $V_h \subset H(div,\Omega)$ and $Q_h \subset L^2(\Omega)/R$ be finite dimensional spaces and $V_h^0 = V_h \cap H_0(div,\Omega)$. The numerical approximation problem associated with (3.31) is to find $(u_h, p_h) \in v_h \times Q_h$ such that $u_h \cdot n = g_h$ on $\partial\Omega$, where $g_h = g_{0,h} \cdot n$ on $\partial\Omega$ and $g_{0,h} = \sum_{e \in \{\partial K \bigcap \partial\Omega, K \in \tau_h\}} (\int_e g ds) N_e$, $N_e \in V_h$, is corresponding basis function to edge e, and

$$\begin{aligned} ((\lambda k)^{-1} u_h, v_h) + (div v_h, p_h) &= 0 \quad \forall v_h \in V_h^0 \\ (div u_h, q_h) &= 0 \quad \forall q_h \in Q_h. \end{aligned} \tag{3.33}$$

One can define a linear operator $B_h : V_h^0 \to Q_h'$ by $(B_h u_h, q_h) = b(u_h, q_h)$.

Suppose that the following conditions are satisfied

$$a(u_h, u_h) \quad \text{is} \quad \ker B_h - \text{coercive} \tag{3.34}$$

$$\inf_{q_h \in Q_h} \sup_{v_h \in V_h} \frac{b(v_h, q_h)}{\|v_h\|_{H(div,\Omega)} \|q_h\|_{L^2(\Omega)}} \geq C. \tag{3.35}$$

Then the following approximation property follows (see e.g., [19]).

Lemma 3.2.4. *If (u,p) and (u_h,p_h) respectively solve the problem (3.31) and (3.33) and the conditions (3.34) and (3.35) hold, then*

$$\|u - u_h\|_{H(div,\Omega)} + \|p - p_h\|_{0,\Omega} \leq \inf_{\substack{v_h \in V_h \\ v_h - g_{0,h} \in V_h^0}} \|u - v_h\|_{H(div,\Omega)} + \inf_{q_h \in Q_h} \|p - q_h\|_{0,\Omega}. \tag{3.36}$$

If we assume that $V_h = RT_0$, the lowest Raviart-Thomas space, and $Q_h = \bigoplus_K P_0(K)$, the piecewise constant space, then the standard mixed finite method theory [19] implies the following estimate for scale-separation problem with physical scale length ϵ.

Proposition 3.2.5. *If $V_h = RT_0$ and $Q_h = \bigoplus_K P_0(K)$ and $f \in H^1$, then*

$$\|u_\epsilon - u_h\|_{H(div,\Omega)} + \|p_\epsilon - p_h\|_{0,\Omega} \leq C\frac{h}{\epsilon}\|f\|_{0,\Omega} + Ch|f|_{1,\Omega} + C\epsilon\|g\|_{-\frac{1}{2},\partial\Omega}. \tag{3.37}$$

Remark 3.2.2. *In the Proposition 3.2.5, we have used the regularity estimate, i.e.,*

$$|u_\epsilon|_{1,\Omega} \leq C\epsilon^{-1}\|f\|_{1,\Omega}.$$

Since ϵ is very small in practice, Proposition 3.2.5 implies that traditional mixed finite element does not work well for the multiscale elliptic problem.

Local Mixed MsFEM . We would like to follow the local mixed MsFEM in [22] to discuss the scale-separation problem

$$\begin{aligned} &div\lambda(x)k^\epsilon(x)\nabla p_\epsilon = 0 \quad \text{in} \quad \Omega \\ &\lambda(x)k^\epsilon(x)\nabla p_\epsilon \cdot n = g(x) \quad \text{on} \quad \partial\Omega, \end{aligned} \tag{3.38}$$

where $k^\epsilon = k(x/\epsilon)$ is ϵ-periodic and $\Omega \subset R^d$. Owing to multiscale expansion and identifying the powers of ϵ, we can get the following homogenization equation

$$\begin{aligned} &div\lambda(x)k^*\nabla p_0 = 0 \quad \text{in} \quad \Omega \\ &\lambda(x)k^*\nabla p_0 \cdot n = g(x) \quad \text{on} \quad \partial\Omega, \end{aligned} \tag{3.39}$$

where k^* is the homogenized matrix and is defined previously.

For further analysis, we define first order corrector $p_{\epsilon,1} = p_0 + \epsilon\chi\nabla p_0$, and the boundary corrector $\theta_\epsilon \in H^1/R$, which solves

$$div\lambda(x)k^\epsilon(x)\nabla\theta_\epsilon = 0 \quad \text{in} \quad \Omega \tag{3.40}$$

$$\lambda(x)k^\epsilon(x)\nabla\theta_\epsilon \cdot n = \lambda\frac{\partial}{\partial x_j}(\alpha^k_{ij}(\frac{x}{\epsilon})\frac{\partial p_0}{\partial x_k}) \cdot n_i \quad \text{on} \quad \partial\Omega, \tag{3.41}$$

where $n = \{n_1, \cdots, n_d\}$ is the unit outer normal to $\partial\Omega$ and α^k_{ij} is zero mean skewsymmetric matrix such that $\frac{\partial \alpha_{ij}^k}{\partial y_j} = k_{il} + k_{ij}\frac{\partial \chi_l}{\partial y_j} - k_{il}^*$. The following estimate is derived in [22].

Lemma 3.2.6. *Assume that* $p_0 \in H^2(\Omega) \cap W^{1,\infty}(\Omega)$, *then*

(1) $\|\nabla p_\epsilon - \nabla(p_{\epsilon,1} + \epsilon\theta_\epsilon)\|_{0,\Omega} \le C\epsilon\|\lambda\|_{0,\infty}(|p_0|_{2,\Omega} + |p_0|_{1,\Omega})$;

(2) $\|\epsilon\nabla\theta_\epsilon\|_{0,\Omega} \le C\epsilon\|\lambda\|_{0,\infty}\|p_0\|_{2,\Omega} + C\sqrt{\epsilon|\partial\Omega|}\|\lambda\|_{0,\infty}|p_0|_{1,\infty,\Omega}$.

The similar result holds for the solution of single-phase flow equation by p_ϵ^{sp}, i.e.,

$$\|\nabla p_\epsilon^{sp} - \nabla(p_{\epsilon,1}^{sp} + \epsilon\theta_\epsilon)\|_{0,\Omega} \le C\epsilon(|p_0^{sp}|_{2,\Omega} + |p_0^{sp}|_{1,\Omega})$$

and

$$\|\epsilon\nabla\theta_\epsilon^{sp}\|_{0,\Omega} \le C\epsilon\|p_0^{sp}\|_{2,\Omega} + C\sqrt{\epsilon|\partial\Omega|}|p_0^{sp}|_{1,\infty,\Omega}.$$

For Mixed MsFEM, one need to solve local problems to construct basis functions for velocity. Following Chen and Hou [22] (see also [9]), one can construct multiscale basis functions for velocity in each coarse block K

$$\begin{aligned} div(k(x)\nabla w_i^K) &= \frac{1}{|K|} \quad \text{in} \quad K \\ k(x)\nabla w_i^K n^K &= \begin{cases} g_i^K & \text{on } e_i^K \\ 0 & \text{else,} \end{cases} \end{aligned} \tag{3.42}$$

where e_i^K's are the edges of K and $g_i^K = \frac{1}{|e_i^K|}$ in local mixed MsFEM proposed in [22]. Then, we can define the finite dimensional space for velocity by

$$\begin{aligned} V_h &= \bigoplus_K \{\psi_i^K\}, \\ V_h^0 &= V_h \cap H_0(div, \Omega), \end{aligned}$$

where $\psi_i^K = k(x)\nabla w_i^K$.

Let $Q_h = \bigoplus_K P_0(K) \cap L^2(\Omega)/R$, a set of piece-wise constant functions. We state the convergence theorem for scale separation problem as following.

Theorem 3.2.7. *Let* $(u_\epsilon, p_\epsilon) \in H(div, \Omega) \times L^2(\Omega)/R$ *solve problem (3.31) and* $(u_h, p_h) \in V_h \times Q_h$ *solve the discrete variation problem (3.33). If the homogenized solution* $p_0 \in H^2(\Omega) \cap W^{1,\infty}(\Omega)$*, then*

$$\begin{aligned} &\|u_\epsilon - u_h\|_{H(div,\Omega)} + \|p_\epsilon - p_h\|_{0,\Omega} \\ &\leq C_1(p_0, \lambda)\epsilon + C_2(p_0, \lambda, g)h + C_3(p_0, \lambda)\sqrt{\epsilon h} + C_4(p_0, \lambda)\sqrt{\frac{\epsilon}{h}}, \end{aligned} \tag{3.43}$$

where the coefficients are defined in (3.54), (3.58) ,(3.56) and (3.57) respectively.

We give a brief proof for the sake of completeness (also refer to [22]).

Under the above finite element spaces, the well-posedness of the discrete problem is easily verified [22], that is, (3.34) and (3.35) are true for the local mixed MsFEM spaces.

In order to make the error analysis we only need to make an estimation of right-hand in (3.36), in which the second part is easy to estimate. We have

Proposition 3.2.8. *Let* p_ϵ *and* p_h *be the solution of pressure in (3.31) and (3.33) respectively, then*

$$\inf_{q_h \in Q_h} \|p_\epsilon - q_h\|_0 \leq Ch\|g\|_{H^{-\frac{1}{2}}(\partial\Omega)} \tag{3.44}$$

Proof. Define $\bar{q}_h = \frac{1}{|K|}\int_K p_\epsilon dx$ in each coarse block K. Then we apply Poincaré inequality (2.9) and regularity estimation of elliptic equations, the conclusion follows immediately. That is

$$\inf_{q_h \in Q_h} \|p_\epsilon - q_h\|_0 \leq \|p_\epsilon - \bar{q}_h\|_0 \leq Ch\|\nabla p_\epsilon\|_0 \leq Ch\|g\|_{H^{-\frac{1}{2}}(\partial\Omega)}$$

□

We define the multiscale interpolation operator $\Pi_h : H(div, \Omega) \longrightarrow V_h$ by

$$\Pi_h u|_K = \left(\int_{e_i^K} u \cdot n^K ds\right)\psi_i^K. \tag{3.45}$$

Let $RT_0 = span\{R_i^K, i = 1, 2, \cdots, n; K \in \tau_h\}$ be the lowest order Raviart-Thomas space and define the interpolation operator $P_h : H(div, \Omega) \longrightarrow RT_0$ by

$$P_h u|_K = (\int_{e_i^K} u \cdot n^K ds) R_i^K. \tag{3.46}$$

It is easy to check that $div \Pi_h u = div P_h u$ and $\Pi_h u \cdot n^K = P_h u \cdot n^K$.

Right now we still need to estimate the first term in the righthand of (3.36). The basic idea is to choose a particular u_h approximating u_ϵ very well. Let the homogenized velocity $u_0 = \lambda k^* \nabla p_0$ and choose $t_h|_K = \Pi_h u_0$ and so we have that $t_h - g_h \in V_h^0$, where $g_h = \sum_{e \in \partial\Omega} (\int_e g ds) \psi_i^K$. Consequently, it remains to estimate $\|u_\epsilon - t_h\|_{H(div,\Omega)}$. From the definition of t_h, an easy calculation gives rise to $div(t_h|_K) = 0$ and $div u_\epsilon = 0$. Therefore, we have

$$\|div u_\epsilon - div t_h\|_{0,\Omega} = 0. \tag{3.47}$$

The next step is to estimate $\|u_\epsilon - t_h\|_{0,\Omega}$. The nature idea is to use homogenization technique. We set $w_\epsilon^K = \alpha_i^K w_i^K$,where $\alpha_i^K = \int_{e_i^K} u_0 \cdot n^K ds$ and then $t_h = k^\epsilon \nabla w_\epsilon^K$ and $div k^\epsilon \nabla w_\epsilon^K = div P_h u_0 = 0$ in K, where $w_\epsilon^K \in H^1(K)/R$ satisfies the following equation

$$\begin{aligned} div k^\epsilon \nabla w_\epsilon^K &= 0 \quad \text{in} \quad K & (3.48) \\ k^\epsilon \nabla w_\epsilon^K \cdot n^K &= P_h u_0 \cdot n^K \quad \text{on} \quad e_i^K. & (3.49) \end{aligned}$$

Let w_0^K be the solution of the corresponding homogenization equation, that is

$$\begin{aligned} div k^* \nabla w_0^K &= 0 \quad \text{in} \quad K & (3.50) \\ k^* \nabla w_0^K \cdot n^K &= P_h u_0 \cdot n^K \quad \text{on} \quad e_i^K. & (3.51) \end{aligned}$$

Let $w_1^K = w_0^K + \epsilon \chi \nabla w_0^K$. To estimate $\|u_\epsilon - t_h\|_{0,\Omega}$, we need the following lemma.

Lemma 3.2.9. *Let p_1^ϵ and w_1^K be defined in the above, then*

$$\begin{aligned}
|w_0^K - p_0|_{1,K} &\le Ch\|\lambda^{-1}-1\|_{0,\infty,K}\|\lambda\|_{1,\infty,K}\|p_0\|_{2,K} \\
|p_1^\epsilon - w_1^K|_{1,K} &\le C(h\|\lambda^{-1}-1\|_{0,\infty,K}+\epsilon)\|\lambda\|_{1,\infty,K}\|p_0\|_{2,K} \\
|w_0^K|_{1,\infty,K} &\le Ch^{-\frac{d}{2}+1}\|\lambda\|_{1,\infty,K}\|p_0\|_{2,K} + C\|\lambda\|_{0,\infty,K}|p_0|_{1,\infty,K}.
\end{aligned} \tag{3.52}$$

Proof. Owing to $k^*\nabla w_0^K = P_h u_0 \in L^\infty(K)$, we get that $w_0^K \in H^2(K)\cap W^{1,\infty}(K)$. Applying the interpolation estimate of Raviart-Thomas finite element, we obtain

$$\begin{aligned}
|w_0^K - p_0|_{1,K} &= \|(k^*)^{-1}P_h u_0 - (\lambda k^*)^{-1}u_0\|_{0,K} \\
&\le C\|\lambda^{-1}-1\|_{0,\infty,K}\|P_h u_0 - u_0\|_{0,K} \\
&\le Ch\|\lambda^{-1}-1\|_{0,\infty,K}|u_0|_{1,K} \\
&\le Ch\|\lambda^{-1}-1\|_{0,\infty,K}\|\lambda\|_{1,\infty,K}\|p_0\|_{2,K}.
\end{aligned}$$

Since $\nabla w_0^K = (k^*)^{-1}P_h u_0$ and P_h is bounded, we get

$$\begin{aligned}
|w_0^K|_{1,K} &\le C|\lambda|_{0,\infty,K}|p_0|_{1,K} \\
|w_0^K|_{2,K} &\le C\|\lambda\|_{1,\infty,K}|p_0|_{2,K}.
\end{aligned}$$

Using the above estimations, we obtain

$$\begin{aligned}
|p_1^\epsilon - w_1^K|_{1,K} &\le |p_0 - w_0^K|_{1,K} + \|(\nabla_y\cdot\chi)\nabla(p_0 - w_0^K)\|_{0,k} + \epsilon\|\chi(\nabla^2 p_0 - \nabla^2 w_0^K)\|_{0,K} \\
&\le Ch\|\lambda^{-1}-1\|_{0,\infty,K}\|\lambda\|_{1,\infty,K}\|p_0\|_{2,K} + C\epsilon\|\lambda\|_{1,\infty,K}|p_0|_{2,K}.
\end{aligned}$$

As for the last inequality in (3.52), we invoke the inverse inequality of finite elements and then we get that

$$
\begin{aligned}
|w_0^K|_{1,\infty,K} &\leq C\|P_h u_0 - < u_0 >_K \|_{0,\infty,K} + C\| < u_0 >_K \|_{0,\infty,K} \\
&\leq Ch^{-\frac{d}{2}}\|P_h u_0 - < u_0 >_K \|_{0,K} + C\| < u_0 >_K \|_{0,\infty,K} \\
&\leq Ch^{-\frac{d}{2}+1}\|\lambda\|_{1,\infty,K}\|p_0\|_{2,K} + C\|\lambda\|_{0,\infty,K}|p_0|_{1,\infty,K}
\end{aligned}
$$

□

Applying the definitions of u_ϵ and t_h and the Lemma 3.2.9, we estimate $\|u_\epsilon - t_h\|_{0,\Omega}$ in the following way.

$$
\begin{aligned}
&\|u_\epsilon - t_h\|_{0,K} \\
&\leq C\|\lambda - 1\|_{0,\infty,K}(\|\nabla p_\epsilon - \nabla w_\epsilon^K\|_{0,K}) \\
&\leq C\|\lambda - 1\|_{0,\infty,K}(\|\nabla p_\epsilon - \nabla p_1^\epsilon\|_{0,K} + \|\nabla p_1^\epsilon - \nabla w_1^K\|_{0,K} + \|\nabla w_1^K - \nabla w_\epsilon^K\|_{0,K}) \\
&\leq C\|\lambda - 1\|_{0,\infty,K}\{\epsilon(\|\lambda\|_{0,\infty,K}\|p_0\|_{2,K} + \|w_0^K\|_{2,K}) + \\
&\sqrt{\epsilon h^{d-1}}(\|\lambda\|_{0,\infty,K}|p_0|_{1,\infty,K} + |w_0^K|_{1,\infty,K}) + (h\|\lambda^{-1} - 1\|_{0,\infty,K} + \epsilon)\|\lambda\|_{1,\infty,K}\|p_0\|_{2,K}\} \\
&\leq C_{K,1}(p_0,\lambda)\epsilon + C_{K,2}(p_0,\lambda)h + C_{K,3}(p_0,\lambda)\sqrt{\epsilon h} + C_{K,4}(p_0,\lambda)\sqrt{\epsilon h^{d-1}},
\end{aligned}
$$

where we have used Lemma 3.2.6 and the following notations

$$
\begin{aligned}
C_{K,1}(p_0,\lambda) &= C\|\lambda - 1\|_{0,\infty,K}\|\lambda\|_{1,\infty,K}\|p_0\|_{2,K} \\
C_{K,2}(p_0,\lambda) &= C\|\lambda - 1\|_{0,\infty,K}\|\lambda^{-1} - 1\|_{0,\infty,K}\|\lambda\|_{1,\infty,K}\|p_0\|_{2,K} \\
C_{K,3}(p_0,\lambda) &= C\|\lambda - 1\|_{0,\infty,K}\|\lambda\|_{1,\infty,K}\|p_0\|_{2,K} \\
C_{K,4}(p_0,\lambda) &= C\|\lambda - 1\|_{0,\infty,K}(1 + \|\lambda\|_{0,\infty,K})\|p_0\|_{1,\infty,K}.
\end{aligned}
$$

Therefore, taking summation all over K, we have

$$
\|u_\epsilon - t_h\|_{0,\Omega} \leq C_1(p_0,\lambda)\epsilon + \tilde{C}_2(p_0,\lambda)h + C_3(p_0,\lambda)\sqrt{\epsilon h} + C_4(p_0,\lambda)\sqrt{\frac{\epsilon}{h}}, \tag{3.53}
$$

where we have used the fact that the family of the mesh is quasi-uniform, and the notations are

$$C_1(p_0,\lambda) = C\|\lambda-1\|_{0,\infty,\Omega}\|\lambda\|_{1,\infty,\Omega}\|p_0\|_{2,\Omega} \tag{3.54}$$

$$\tilde{C}_2(p_0,\lambda) = C\|\lambda-1\|_{0,\infty,\Omega}\|\lambda^{-1}-1\|_{0,\infty,\Omega}\|\lambda\|_{1,\infty,\Omega}\|p_0\|_{2,\Omega} \tag{3.55}$$

$$C_3(p_0,\lambda) = C\|\lambda-1\|_{0,\infty,\Omega}\|\lambda\|_{1,\infty,\Omega}\|p_0\|_{2,\Omega} \tag{3.56}$$

$$C_4(p_0,\lambda) = C\|\lambda-1\|_{0,\infty,\Omega}(1+\|\lambda\|_{0,\infty,\Omega})\|p_0\|_{1,\infty,\Omega}. \tag{3.57}$$

Using (3.47) and applying the Proposition 3.2.8, we immediately get

$$\begin{aligned}&\|u_\epsilon-u_h\|_{H(div,\Omega)}+\|p_\epsilon-p_h\|_{0,\Omega}\\ &\le C_1(p_0,\lambda)\epsilon+C_2(p_0,\lambda,g)h+C_3(p_0,\lambda)\sqrt{\epsilon h}+C_4(p_0,\lambda)\sqrt{\frac{\epsilon}{h}},\end{aligned}$$

where

$$C_2(p_0,\lambda,g)=\tilde{C}_2(p_0,\lambda)+C\|g\|_{-\frac{1}{2},\partial\Omega}. \tag{3.58}$$

This completes the proof.

Remark 3.2.3. *It follows from the above proof that the resonance term $O(\sqrt{\frac{\epsilon}{h}})$ comes from the terms $|p_0|_{1,\infty,K}$. If the p_0 can be exactly solved by some finite element method, then we can use inverse inequality, i.e.,*

$$\|p_0\|_{1,\infty,K}\le Ch^{1-\frac{d}{2}}\|p_0\|_{2,K}$$

and the Lemma 3.2.9 implies that

$$|w_0^K|_{1,\infty,K}\le Ch^{-\frac{d}{2}+1}\|\lambda\|_{1,\infty,K}\|p_0\|_{2,K},$$

so resonance is vanished. Consequently, the convergence rate would be $O(\epsilon+h+\sqrt{\epsilon h})$.

Remark 3.2.4. *From the proof of the convergence theorem, it follows that $\lambda\in W^{1,\infty}(\Omega)$ and $\lambda^{-1}\in L^\infty(\Omega)$.*

Remark 3.2.5. *If over-sampling technique is used to approximate the flux* u_ϵ *(see [22]), then the resonance error* $O(\sqrt{\frac{\epsilon}{h}})$ *can be reduced to* $O(\frac{\epsilon}{h})$.

Mixed MsFEM Employing Single Global Information . We have already reviewed local mixed MsFEM. Now we propose a mixed MsFEM that employs single-phase flow information. Suppose that p^{sp} solves the single-phase flow equation. We set $b_i^K = (k\nabla p^{sp}|_{e_i^K}) \cdot n^K$ and assume that b_i^K is uniformly bounded. Then the new basis functions for velocity is constructed by solving problems (3.42) with Neumann boundary condition $g_i^K = b_i^K/\beta_i^K$, where $\beta_i^K = \int_{e_i^K} k\nabla p^{sp} \cdot n^K ds$. For further analysis, we assume that $\beta_i^K \neq 0$. In general, if $\beta_i^K = 0$ one can use local mixed multiscale finite element basis function to fix this part. Let $\psi_i^K = k(x)\nabla w_i^K$ and the multiscale finite dimensional space V_h^0 for velocity be defined by

$$\begin{aligned} V_h &:= \bigoplus_K \{\psi_i^K\} \subset H(div, \Omega), \\ V_h^0 &:= V_h \cap H_0(div, \Omega). \end{aligned}$$

First, we will show that the resulting multiscale finite element solution for velocity is exact for single-phase flow (i.e., $\lambda(x) = 1$). Let $v_h|_K = \beta_i^K \psi_i^K$, then β_i^K is the interpolation value of the fine scale solution. Furthermore, a direct calculation yields $(v_h|_{e_i^K}) \cdot n^K = k\nabla p^{sp} \cdot n^K$. Because

$$div v_h = \beta_i^K div \psi_i^K = \frac{1}{|K|}\int_{\partial K} k\nabla p^{sp} \cdot n^K ds = \frac{1}{|K|}\int_K div(k\nabla p^{sp} d) = 0,$$

the following equation is obtained immediately

$$div v_h = 0 \quad \text{in} \quad K \tag{3.59}$$

$$v_h \cdot n^K = k\nabla p^{sp} \cdot n^K \quad \text{on} \quad \partial K \tag{3.60}$$

Because $div(k\nabla p^{sp}) = 0$, we get the following proposition.

Proposition 3.2.10. *Let $\beta_i^K = \int_{e_i^K} k\nabla p^{sp} \cdot n^K ds$, then on each coarse block K*

$$k\nabla p^{sp} = \beta_i^K \psi_i^K. \tag{3.61}$$

We re-formulate our assumption for the analysis of mixed multiscale finite element methods. From (3.8), it follows that

$$\|\nabla p - G'(p^{sp})\nabla p^{sp}\|_{0,\Omega} \leq C\delta.$$

Using the fact that k and $\lambda(x)$ are bounded, we have

$$\|\lambda(x)k\nabla p - G'(p^{sp})\lambda(x)k\nabla p^{sp}\|_{0,\Omega} \leq C\delta.$$

Noting that $u = \lambda(x)k\nabla p$ and $u^{sp} = k\nabla p^{sp}$, it follows that there exists a coarse-scale function scalar $A(x)$ such that

$$\|u - A(x)u^{sp}\|_{0,\Omega} \leq \delta. \tag{3.62}$$

Since $A(x)u^{sp}$ approximates u, we assume that it has small divergence,

$$|\int_K div(A(x)u^{sp})dx| \leq C\delta_1 h^2 \tag{3.63}$$

in each K, where δ_1 is a small number. For our analysis, we note that (3.63) gives

$$|\int_{\partial K} A(x)u^{sp} \cdot n^K ds| \leq C\delta_1 h^2. \tag{3.64}$$

We will assume that $A(x) \in C^\gamma$ $(0 < \gamma \leq 1)$. (3.64) can be written as

$$|\sum_i A_i \int_{e_i^K} u^{sp} \cdot n^K ds| \leq C\delta_1 h^2. \tag{3.65}$$

Here A_i's are defined as $A_i = \int_{e_i^K} A(x)u^{sp} \cdot n^K ds / \int_{e_i^K} u^{sp} \cdot n^K ds$, since $\int_{e_i^K} u^{sp} \cdot n^K ds = \beta_i^K \neq 0$. Note that *not* for any $A(x)$, A_i is necessarily a value of $A(x)$ along the edge e_i^K because $u^{sp} \cdot n^K$ can change sign. However, we only need to define $A(x)$ for each edge by its value A_i (e.g., the value of $A(x)$ at the center of edge). Then, for any such $A(x)$, (3.62) is satisfied provided $\delta < h^\gamma$. This can be directly verified. Thus, our main assumption will be (3.62) and (3.65), where $A(x)$ is defined, for example, at the center of each edge e_i^K. We would like to note that from the fact that $div(A(x)u^{sp})$ is small in each K, it follows that $A(x)$, for example, can be taken as an approximation of stream function corresponding to u^{sp}.

Next, we present our error analysis. If inf-sup condition (3.35) is satisfied, then (3.36) hold. We will investigate the inf-sup condition under some specific assumptions for the mixed MsFEM using multiple global information in Section 3.3.3. Based on (3.36), we estimate $\|u - c_i^K \psi_i^K\|_{H(div,\Omega)}$ with appropriate c_i^K.

$$\|u - c_i^K \psi_i^K\|_{H(div,\Omega)} \leq \|u - c_i^K \psi_i^K\|_{0,\Omega} + \|div(c_i^K \psi_i^K)\|_{0,\Omega}. \tag{3.66}$$

Because $div(\psi_i^K) = 1/|K|$, the second term is equal to $\frac{1}{h}\sum_K |\sum_i c_i^K|$. Next, we choose $c_i^K = A_i \beta_i^K$. Then, noticing that $\beta_i^K = \int_{e_i^K} u^{sp} \cdot n^K ds$, we have from (3.65)

$$|\sum_i c_i^K| \leq C\delta_1 h^2$$

for each K. Consequently, for the second term on the right hand side of (3.66), we have

$$\|div(c_i^K \psi_i^K)\|_{0,\Omega} \leq C\delta_1.$$

For the estimation of the first term on the right hand side of (3.66), we have

$$\begin{aligned}
&\|u - c_i^K \psi_i^K\|_{0,K} \\
&\leq \|u - A(x)u^{sp}\|_{0,K} + \|A(x)u^{sp} - c_i^K \psi_i^K\|_{0,K} \\
&\leq \|u - A(x)u^{sp}\|_{0,K} + \|(A(x) - \overline{A}_K)u^{sp}\|_{0,K} + \|\overline{A}_K u^{sp} - A_i \beta_i^K \psi_i^K\|_{0,K} \qquad (3.67) \\
&\leq \|u - A(x)u^{sp}\|_{0,K} + C\|A(x) - \overline{A}_K\|_{\infty,K}\|u^{sp}\|_{0,K} + \|(\overline{A}_K - A_i)\beta_i^K \psi_i^K\|_{0,K} \\
&\leq \|u - A(x)u^{sp}\|_{0,K} + C\|A(x) - \overline{A}_K\|_{\infty,K}\|u^{sp}\|_{0,K} + |\overline{A}_K - A_i|h,
\end{aligned}$$

where $\overline{A}_K$ is the mean of A_i. Here, we have taken into account that $|\beta_i| \leq Ch$ and used Proposition 3.2.10. Summing (3.67) over all K and taking into account $A(x) \in C^\gamma$, we have

$$\|u - c_i \psi_i^K\|_{0,\Omega} \leq C\delta + Ch^\gamma.$$

Thus, we have the following estimate

$$\|u - c_i^K \psi_i^K\|_{H(div,\Omega)} \leq C\delta + C\delta_1 + Ch^\gamma.$$

According to (3.36), for those K, $K \bigcap \partial\Omega \neq 0$, we will adjust proper c_i^K such that $c_i^K \psi_i^K - g_{0,h} \in V_h^0$, but this adjustment will not affect our convergence rate. As for pressure approximation, choosing $q_h = \langle p \rangle_K$ in each K, where $\langle \cdot \rangle_K$ denotes the average over K, we can obtain an estimate for the second term on right hand side of (3.36) by Ch.

Theorem 3.2.11. *Assume (3.62) and (3.65) and $A(x) \in C^\gamma$, $0 < \gamma \leq 1$. Let (u, p) and (u_h, p_h) respectively solve the problem (3.31) and (3.33) with single-phase flow based mixed multiscale finite element, then*

$$\|u - u_h\|_{H(div,\Omega)} + \|p - p_h\|_{0,\Omega} \leq C\delta + C\delta_1 + Ch^\gamma. \qquad (3.68)$$

Remark 3.2.6. *By lemma 5.3 in [3], no inf-sup condition is assumed and it follows* *that*

$$\|u - u_h\|_{0,\Omega} \leq C\delta + Ch^{\gamma}.$$

In the following, we will use the mixed MsFEM using single phase flow information to solve a scale separation (ϵ-scale periodic) problem (3.38).

For the error analysis, we assume (3.23):

$$|p_0 - G(p_0^{sp})|_{1,\Omega} \leq C\delta_0. \tag{3.69}$$

As we showed previously that if $k = k(x/\epsilon)$, this assumption implies the assumption (3.8), which implies in its turn (3.62). (3.69) is equivalent to

$$\|\nabla p_0 - G'(p_0^{sp})\nabla p_0^{sp}\|_{0,\Omega} \leq C\delta_0.$$

Because k^* and $\lambda(x)$ are bounded, we have

$$\|u_0 - A(x)u_0^{sp}\|_{0,\Omega} \leq C\delta_0. \tag{3.70}$$

(3.70) is the main assumption for the case with scale separation along with the assumption (cf. (3.65))

$$|\sum_i A_i \int_{e_i^K} u_0^{sp} \cdot n^K ds| \leq C\delta_1 h^2. \tag{3.71}$$

Similar to the case of Galerkin method, one can derive the convergence rate for mixed multiscale finite element method. In this case, the convergence rate does not contain the resonance error and the following theorem holds.

Theorem 3.2.12. *Assume (3.70), (3.71), $A(x) \in C^{\gamma}$ $(0 < \gamma \leq 1)$, $|p_0|_{2,\Omega} + |p_0^{sp}|_{2,\Omega}$ and $\|\nabla p_0\|_{\infty,\Omega} + \|\nabla p_0^{sp}\|_{\infty,\Omega}$ are bounded. Let (u_ϵ, p_ϵ) and (u_h, p_h) respectively solve the problem (3.31) and (3.33) with single-phase flow based mixed multiscale finite element, then*

$$\|u_\epsilon - u_h\|_{H(div,\Omega)} + \|p_\epsilon - p_h\|_{0,\Omega} \leq C\delta_0 + C\delta_1 + C\sqrt{\epsilon} + Ch^{\gamma}. \tag{3.72}$$

Proof. Set

$$\tilde{u}_\epsilon = \lambda k^\epsilon (I + \nabla_y \chi)\nabla p_0 + \epsilon\lambda k^\epsilon \nabla^2 p_0 \chi + \epsilon\lambda k^\epsilon \nabla\theta_\epsilon$$

and

$$\tilde{u}_\epsilon^{sp} = k^\epsilon (I + \nabla_y \chi)\nabla p_0^{sp} + \epsilon k^\epsilon \nabla^2 p_0^{sp} \chi + \epsilon k^\epsilon \nabla\theta_\epsilon^{sp}$$

where χ, θ_ϵ and θ_ϵ^{sp} are defined in the part of local mixed MsFEM.

We define c_i^K as the same as in the case continuum scale. Hence we have

$$\|u_\epsilon - c_i^K \psi_i^K\|_{H(div,\Omega)} \le \|u_\epsilon - c_i^K \psi_i^K\|_{0,\Omega} + C\delta_1. \tag{3.73}$$

As for the first term on the right hand side of (3.73), we have

$$\begin{aligned} \|u_\epsilon - c_i^K \psi_i^K\|_{0,\Omega} &\le \|u_\epsilon - A(x)u_\epsilon^{sp}\|_{0,\Omega} + \|A(x)u_\epsilon^{sp} - c_i^K \psi_i^K\|_{0,\Omega} \\ &\le \|u_\epsilon - A(x)u_\epsilon^{sp}\|_{0,\Omega} + Ch^\lambda, \end{aligned} \tag{3.74}$$

where we used the same argument as in the case of continuum scale for the second term in right hand side.

A straightforward computation gives rise to

$$\begin{aligned} \|u_\epsilon - A(x)u_\epsilon^{sp}\|_{0,\Omega} &\le \|u_\epsilon - \tilde{u}_\epsilon\|_{0,\Omega} + \|\tilde{u}_\epsilon - A(x)\tilde{u}_\epsilon^{sp}\|_{0,\Omega} + \|A(x)\tilde{u}_\epsilon^{sp} - A(x)u_\epsilon^{sp}\|_{0,\Omega} \\ &\le (C\epsilon|p_0|_{2,\Omega} + C\sqrt{\epsilon}|p_0|_{1,\infty,\Omega}) + C\delta_0 + (C\epsilon|p_0^{sp}|_{2,\Omega} \\ &+ C\sqrt{\epsilon}|p_0^{sp}|_{1,\infty,\Omega}) \\ &\le C\epsilon(|p_0|_{2,\Omega} + |p_0^{sp}|_{2,\Omega}) + C\sqrt{\epsilon}(|p_0|_{1,\infty,\Omega} + |p_0^{sp}|_{1,\infty,\Omega}) + C\delta_0, \end{aligned} \tag{3.75}$$

where we have used Theorem 3.1 in [22].

If we choose $q_h|_K =< p_\epsilon >_K$, then

$$\|p_\epsilon - q_h\|_{0,\Omega} \le Ch.$$

Invoking Lemma 3.2.4, (3.73), (3.74) and (3.75), Theorem 3.2.12 follows immediately.

□

Remark 3.2.7. *Comparing Theorem 3.2.7 and Theorem 3.2.12 , we see that the resonance error* $\sqrt{\frac{\epsilon}{h}}$ *is removed by applying the mixed MsFEM using limited global information.*

3.3. Multiscale Methods Using Multiple Global Information for Elliptic Equations

In this section, we consider some multiscale methods using multiple global fields. First we introduce partition of unity method (PUM) and investigate the global multiscale method based on PUM. Second we introduce multiscale finite element method based on harmonic coordinates proposed in [53]. Third we propose a mixed MsFEM using multiple global fields.

3.3.1. PUM Using Multiple Global Information

Partition of unity (PUM) method is also a multiscale numerical approach. It was first introduced in [14] to obtain an accurate numerical solution of second order elliptic equations with rough coefficients. Based on this idea, the approach is further elaborated on in [49], where it is named by partition of unity methods (PUM). The main idea of PUM is to find an accurate approximation in each "patch" and then use partition of unity functions to "paste" those patch approximations together. If one knows more information about solutions of partial differential equations and enrich this information into patch approximation, then an accurate approximation of numerical solution can be obtained. Motivated by this idea, we have designed a PUM using global information to solve elliptic and evolution equations where the coefficients have continuum scales. Standard localized multiscale methods or upscaling techniques are not very suitable for these problems because these problems do not possess scale separation and homogenization techniques are not applicable. For our analysis, we use the fact that the solutions of partial differential equations smoothly depend on certain global fields (defined over the entire region), which carry the information on the small scale structure of the solution. These global fields are used to construct shape functions in each patch. This is not a trivial extension of MsFEM that employs limited global information because the construction of basis function in every "patch" are different from that global MsFEM discussed previously, i.e., basis functions in each "patch" are the span of multiplication of partial unity functions and global fields. Our analysis shows that the convergence rate of this PUM in H^1 for continuum scale problems is $O(h^\alpha)$ (h is the size of coarse meshes), which is free of resonance errors as global MsFEM and global mixed MsFEM. This method shares some similarities with the recent work in [53] and [52].

Introduction to PUM . In this subsection, we will apply PUM to the following model elliptic equation

$$\begin{aligned} -div k(x)\nabla p(x) &= f(x) \quad \text{in} \quad \Omega \\ p(x) &= 0 \quad \text{on} \quad \partial\Omega. \end{aligned} \tag{3.76}$$

The numerical variational formulation is to find $p_h \in S_h \subset H_0^1(\Omega)$ such that

$$a(p_h, v_h) = (f, v_h) \quad \forall v_h \in S_h, \tag{3.77}$$

where $a(p_h, v_h) = (k\nabla p_h, \nabla v_h)$.

The definition of PUM in [49] implies that there are three ingredients for PUM. The first one is patch family $\{\omega_i\}$, which a finite open cover for the domain Ω. The second one is the partition of unity $\{\varphi_i\}$ subordinate to $\{\omega_i\}$ satisfying

$$\begin{aligned} supp\varphi_i \subset closure(\omega_i) \quad \forall i, \\ \sum_i \varphi_i(x) = 1, \quad \forall x \in \Omega, \\ \|\varphi_i\|_\infty \leq C_\infty, \quad \forall i, \\ \|\nabla\varphi_i\|_\infty \leq \frac{C_G}{diam(\omega_i)}, \quad \forall i. \end{aligned} \tag{3.78}$$

The third one is the family of local approximation spaces $\{V_i\}$ corresponding to the patch family $\{\omega_i\}$. A finite approximation space S_h^{GM} for the problem (3.77) can be defined by

$$S_h^{GM} := \varphi_i V_i = \{\varphi_i v_i | \;\; v_i \in V_i\} \subset H_0^1(\Omega). \tag{3.79}$$

For better accuracy, we have a modified definition for the finite approximation space S_h^{GM}:

$$S_h^{MGM} := \{a_j\psi_j + \varphi_i v_i | \;\; \psi_j \in S_h^k, v_i \in V_i\} \subset H_0^1(\Omega). \tag{3.80}$$

For example, the modified space S_h^{MGM} is used in [58]. The first term recovers some standard FEM information like polynomial properties. The second term captures the information from each local approximation. For simplicity of our mathematical analysis, we use the regular approximation space S_h^{GM} in this section.

Let $E(\omega_i) := \{p : \int_{\omega_i} k\nabla p \cdot \nabla p dx < \infty\}$ be the energy space on the patch ω_i and for every i, $V_i = span\{\xi_{ik} \in E(\omega_i) : \quad k = 1 \cdots m(k)\}$, then the weak formulation is to seek $p_h = \sum_{i,k} c_{ik}\varphi_i\xi_{ik}$ so that (3.77) holds. The problem reduces to the linear system

$$\mathbf{Ac} = F, \tag{3.81}$$

where the components of the stiffness matrix $\mathbf{A}$ are

$$A(l, s; i, k) = a(\varphi_l\xi_{ls}, \varphi_i\xi_{ik}) \quad (l, i = 1 \cdots N; s = 1 \cdots m(l); k = 1 \cdots m(i)), \tag{3.82}$$

and the element for the load vector F is

$$F(l,s) = (f, \varphi_l \xi_{ls})_{\omega_l}. \tag{3.83}$$

Let ξ_i^p approximate p on each patch ω_i and satisfy that

$$\begin{aligned} \|p - \xi_i^p\|_{\omega_i,0} &\leq \epsilon_1(i), & (3.84)\\ |p - \xi_i^p|_{\omega_i,1} &\leq \epsilon_2(i). & (3.85)\end{aligned}$$

The basic result for convergence analysis of PUM (or GFEM) is as following.

Lemma 3.3.1. *[49] Assume that the local approximation satisfy (3.84) and (3.85). Let ξ^p be the GFEM approximation to p such that $\xi^p = \varphi_i \xi_i^p$, then*

$$\begin{aligned} \|p - \xi^p\|_{0,\Omega} &\leq \sqrt{M} C_\infty (\sum_i \epsilon_1^2(i))^{\frac{1}{2}}, & (3.86)\\ |p - \xi^p|_{1,\Omega} &\leq \sqrt{2M}((\frac{C_G}{diam(\omega_i)})^2 \epsilon_1^2(i) + C_\infty^2 \epsilon_2^2(i))^{\frac{1}{2}}, & (3.87)\end{aligned}$$

where M is the overlap index associated with $\{\omega_i\}$. Furthermore, if the local approximation spaces $\{V_i\}$ satisfy the uniform Poincaré property [12], then there exist $\tilde{\xi}_i^p \in V_i$ such that the global approximation $\tilde{\xi}^p = \varphi_i \tilde{\xi}_i^p$ and

$$\begin{aligned} \|p - \tilde{\xi}^p\|_{0,\Omega} &\leq C(diam^2(\omega_i)\epsilon_2^2(i))^{\frac{1}{2}}, & (3.88)\\ \|p - \tilde{\xi}^p\|_{1,\Omega} &\leq C(\sum_i \epsilon_2^2(i))^{\frac{1}{2}}. & (3.89)\end{aligned}$$

Remark 3.3.1. *If the patches ω_i are convex and not too "flat", then the uniform Poincaré property is satisfied and can be described in terms of some geometric conditions of ω_i [13].*

Local Multiscale PUM . Similar to MsFEM in [39], a multiscale basis for PUM is used to approximate to p in each patch ω_i, that is to say, $V_i = span\{\phi_j^{\omega_i}, j =$

$1 \cdots m(i)\}$, where the generalized multiscale basis $\phi_j^{\omega_i}$ solves

$$\begin{cases} -div(k(x)\nabla\phi_j^{\omega_i}) & = & 0 \quad \text{in} \quad \omega_i \\ \phi_j^{\omega_i} & is & linear \quad \text{on} \quad \partial\omega_i \\ \phi_j^{\omega_i}(z_k^{\omega_i}) & = & \delta_{jk}. \end{cases} \tag{3.90}$$

Here the z_k are the vertexes of ω_i.

We would like to make a comparison with MsFEM proposed in [39] and take $k(x) = k(\frac{x}{\epsilon})$ in (3.76). For this ϵ-periodic problem, Lemma (3.3.1) implies the following theorem.

Theorem 3.3.2. *Let p_ϵ and $p_h \in S_h^{GM}$ solve the problem (3.76) and (3.77) respectively in 1-D (one dimension), then*

$$\|p_\epsilon - p_h\|_1 \leq Ch\|f\|_0 \tag{3.91}$$

and

$$\|p_\epsilon - p_h\|_0 \leq Ch^2\|f\|_0. \tag{3.92}$$

Proof. Let $\xi_i^p = p_I$ in each patch ω_i, where p_I is the interpolation of p_ϵ with basis function defined in (3.90). Then we apply the results of [39] and find that ξ_i^u in each patch $\omega_i \subset \Omega \subset R$ such that

$$\begin{aligned} \|p_\epsilon - \xi_i^p\|_{0,\omega_i} &\leq Ch^2\|f\|_{0,\omega_i}, \quad (3.93) \\ |p_\epsilon - \xi_i^p|_{1,\omega_i} &\leq Ch|f\|_{0,\omega_i}. \quad (3.94) \end{aligned}$$

Let $\xi^p = \varphi_i\xi_i^p$ (summation convention is used). Applying the (3.93) and (3.86), we get that

$$\|p_\epsilon - p_h\|_0 \leq C(\sum_i h^4 \|f\|^2_{0,\omega_i})^{\frac{1}{2}}$$
$$\leq Ch^2\|f\|_0.$$

The proof of (3.92) is done. On the other hand, we notice (3.93), (3.94) and (3.86) so that we obtain

$$|p_\epsilon - p_h|_1 \leq C(\sum_i \frac{C_G^2}{h^2} h^4 \|f\|^2_{0,\omega_i} + C_\infty^2 h^2 \|f\|^2_{0,\omega_i})^{\frac{1}{2}}$$
$$\leq Ch\|f\|_0.$$

Owing to the result (3.92) and the above estimation for semi-norm, the result (3.91) follows immediately. □

As for the model problem (3.76) in d-D ($d \geq 2$), we can get the following results.

Theorem 3.3.3. *Let p_ϵ and $p_h \in S_h^{GM}$ solve the problem (3.76) and (3.77) respectively in d-D ($d \geq 2$), then for $h > \epsilon$*

(1). $\|p_\epsilon - p_h\|_1 \leq Ch\|f\|_0 + C\sqrt{\frac{\epsilon}{h}}$;

(2). $\|p_\epsilon - p_h\|_0 \leq Ch^2\|f\|_0 + C\epsilon + C\frac{\epsilon}{h}$.

Proof. Since the proof of d-D for $d > 2$ is the same as the case of 2-D, we show the proof for 2-D for simplicity. Applying the homogenization technique and taking multiscale expansion for solution u_ϵ and generalized multiscale basis, we can get the above results. These techniques are used in [39]. For completeness, we provide a short proof here.

Let p_0 be the homogenization of the solution p_ϵ to our model problem and $p_I^{\omega_i}$ be the interpolant of $p_0|_{\omega_i}$ with the multiscale basis defined in (3.90). Since p_ϵ and

$p_I^{\omega_i}$ have multiscale structure, they have the multiscale expansion on each patch ω_i,

$$p_\epsilon = p_0 + \epsilon p_1 - \epsilon\theta_\epsilon + Rp_\epsilon, \tag{3.95}$$

$$p_I^{\omega_i} = p_{I0}^{\omega_i} + \epsilon p_{I1}^{\omega_i} - \epsilon\theta_{I\epsilon}^{\omega_i} + Rp_I^{\omega_i}. \tag{3.96}$$

From the first order corrector in [50] and the proof of lemma 5.4 in [39], we obtain that

$$\|Rp_\epsilon\|_{1,\omega_i} \leq C\epsilon|p_0|_{2,\omega_i}, \tag{3.97}$$

$$\|Rp_I^{\omega_i}\|_{1,\omega_i} \leq C\epsilon|p_0|_{2,\omega_i}. \tag{3.98}$$

Applying the triangle inequality and the above estimation for remainder terms, we have

$$\|p_\epsilon - p_I^{\omega_i}\|_{1,\omega_i} \leq C\epsilon|p_0|_{2,\omega_i} + \|p_0 - p_{I0}^{\omega_i}\|_{1,\omega_i} + \|\epsilon(p_1 - p_{I1}^{\omega_i})\|_{1,\omega_i} + \|\epsilon(\theta_\epsilon - \theta_{I\epsilon}^{\omega_i})\|_{1,\omega_i}. \tag{3.99}$$

We are going to estimate the last three terms on the right hand of (3.99). In fact, the interpolation with linear FEM implies that

$$\|p_0 - p_{I0}^{\omega_i}\|_{1,\omega_i} \leq Ch|p_0|_{2,\omega_i}. \tag{3.100}$$

By the definition of p_1 and $p_{I1}^{\omega_i}$, we obtain that

$$\|\epsilon(p_1 - p_{I1}^{\omega_i})\|_{1,\omega_1} \leq (Ch + C\epsilon)|p_0|_{2,\omega_i}. \tag{3.101}$$

Invoking the interpolation inequalities in Sobolev spaces, we get that

$$\begin{aligned}\|\epsilon(\theta_\epsilon - \theta_{I\epsilon}^{\omega_i})\|_{1,\omega_i} &\leq \|\epsilon\theta_\epsilon\|_{1,\omega_i} + \|\epsilon\theta_{I\epsilon}^{\omega_i}\|_{1,\omega_i} \\ &\leq C\epsilon\|p_1\|_{\frac{1}{2},\partial\omega_i} + C\epsilon\|p_{I1}\|_{\frac{1}{2},\partial\omega_i} \\ &\leq C\sqrt{\epsilon}\|p_0\|_{2,\omega_i} + C\sqrt{\epsilon h}.\end{aligned} \tag{3.102}$$

Thanks to (3.99), (3.100), (3.101) and (3.102), the following inequality follows immediately

$$\|p_\epsilon - p_I^{\omega_i}\|_{1,\omega_i} \le C(h + \sqrt{\epsilon})\|p_0\|_{2,\omega_i} + C\sqrt{\epsilon h}. \tag{3.103}$$

Define $\tilde{\xi}_i^{p_\epsilon} = p_I^{\omega_i}$ and $\tilde{\xi}^{p_\epsilon} = \varphi_i \tilde{\xi}_i^{p_\epsilon}$. Owing to the best approximation property and (3.89), a straightforward calculation implies that

$$\begin{aligned}
\|p_\epsilon - p_h\|_{1,\Omega} &\le \|p_\epsilon - \tilde{\xi}^{p_\epsilon}\|_{1,\Omega} \\
&\le C\sqrt{\sum_i \|p_\epsilon - p_I^{\omega_i}\|^2_{1,\omega_i}} \\
&\le Ch\|f\|_{0,\Omega} + C\sqrt{\frac{\epsilon}{h}}.
\end{aligned}$$

This completes the proof of (1). As for the L^2 convergence, we may use the analysis of discrete Green function as in [39] and then use Lemma (3.3.1), we omit the details here. □

Therefore the previous analysis implies that local PUM defined in (3.90) is equivalent to local MsFEM in [39].

PUM Using Multiple Global Information . The multiscale finite element methods in Section 3.2.1 employ information from only one single-phase flow solution. In general, depending on the source term, boundary data, and mobility $\lambda(S)$ (if it contains sharp variations), it might be necessary to use information from multiple global solutions for the computation of accurate two-phase flow solution. The previous multiscale finite element methods can be extended to take into account additional global information. Next, we present an extension of the Galerkin multiscale finite element method that uses the partition of unity method [49].

As the motivation in Section 3.1, we assume that p_1, p_2,..., p_N are the global functions such that $|p - G(p_1, p_2, ..., p_N)|_{1,\Omega}$ is sufficiently small. Here, $p_1, ..., p_N$ can

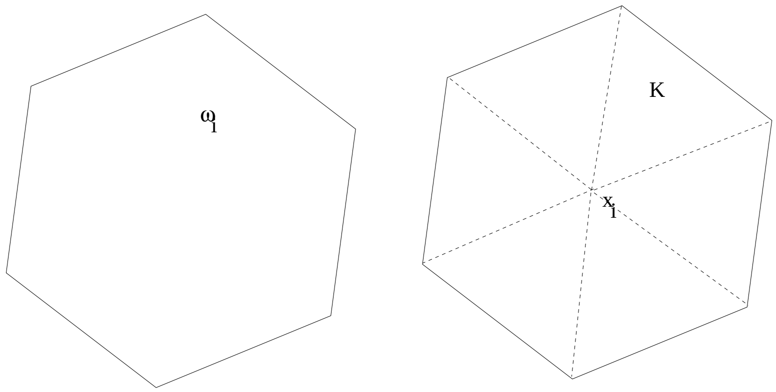

Fig. 3.2. Schematic description of patch

be possible pressure snapshots for different $\lambda(S)$ or pressure fields corresponding to different source terms and/or boundary conditions. Let ω_i be a patch (see Figure 3.2), and define ϕ_i^0 to be piecewise linear basis function in patch ω_i, such that $\phi_i^0(x_j) = \delta_{ij}$. For simplicity of notation, denote $p_1 = 1$. Then, the multiscale finite element method for each patch ω_i is constructed by

$$\psi_{ij} = \phi_i^0 p_j \tag{3.104}$$

where $j = 1, .., N$ and i is the index of nodes (see Figure 3.2). First, we note that in each K, $\sum_i \psi_{ij} = p_j$ is the desired single-phase flow solution.

We will use the following assumption.

There exists a sufficiently smooth scalar valued function $G(\eta)$, $\eta \in R^N$ ($G \in W^{3,\frac{2s}{s-2}}, \quad s > 2$), such that

$$|p - G(p_1, ..., p_N)|_{1,\Omega} \leq C\delta, \tag{3.105}$$

where δ is sufficiently small.

From the stability estimate, we have

$$|p - p_h|_{1,\Omega} \leq |p - G(p_1, ..., p_N)|_{1,\Omega} + |G(p_1, ..., p_N) - c_{ij}\psi_{ij}|_{1,\Omega}, \tag{3.106}$$

where c_{ij} is chosen later. Next, we present the choice of c_{ij} and the estimate for the second term. In each ω_i, we choose c_{ij} as

$$c_{ij} = \frac{\partial G}{\partial p_j}(\overline{p}_1^i, ..., \overline{p}_N^i), \ j \geq 2$$

and

$$c_{i1} = G(\overline{p}_1^i, ..., \overline{p}_N^i) - \frac{\partial G}{\partial p_j}(\overline{p}_1^i, ..., \overline{p}_N^i)\overline{p}_j^i,$$

where $\overline{p}_j^i$ is the average of p_j over ω_i. We note that the following Taylor expansion in each ω_i

$$G(p_1, ..., p_N) = G(\overline{p}_1^i, ..., \overline{p}_N^i) + \frac{\partial G}{\partial p_j}(\overline{p}_1^i, ..., \overline{p}_N^i)(p_j - \overline{p}_j^i) + R_i,$$

where R_i is the remainder given by

$$R_i = \sum_{j,k} \frac{1}{2}\frac{\partial^2 G}{\partial p_k \partial p_j}(\xi_1^i, ..., \xi_N^i)(p_k - \overline{p}_k^i)(p_j - \overline{p}_j^i),$$

where $\xi_k^i = \overline{p}_k^i + \theta^i(p_k - \overline{p}_k^i)$, $0 < \theta^i < 1$. Then, it can be shown that in each ω_i

$$|G(p_1, ..., p_N) - c_{ij}p_j|_{1,\omega_i} \leq |G(\overline{p}_1, ..., \overline{p}_N) + \frac{\partial G}{\partial p_j}(\overline{p}_1, ..., \overline{p}_N)(p_j - \overline{p}_j) - c_{ij}p_j|_{1,\omega_i} + |R|_{1,\omega_i}.$$

The first term on the right hand side is zero because of the choice of c_{ij}. Under the assumption that $p_i \in W^{1,s}(\Omega)$ $(s > 2)$, we have

$$
\begin{aligned}
|R_i|_{1,\omega_i} &\le C \sum_{l,j,k} \|\frac{\partial^3 G}{\partial p_l \partial p_j \partial p_k} \nabla p_l (p_j - \overline{p}_j^i)(p_k - \overline{p}_k^i)\|_{0,\omega_i} \\
&\quad + C \sum_{j,k} \|\frac{\partial^2 G}{\partial p_j \partial p_k} (p_j - \overline{p}_j^i)\nabla p_j + (p_k - \overline{p}_k^i)\nabla p_k\|_{0,\omega_i} \\
&\le C \sum_{j,k,l} h^{2-4/s} |p_j|_{1,s,\omega_i} |p_k|_{1,s,\omega_i} |p_l|_{1,\omega_i} + C \sum_j h^{1-2/s} |p_j|_{1,s,\omega_i} |p_j|_{1,\omega_i} \\
&\le C \sum_{j,k,l} h^{2-4/s} |p_j|_{1,s,\Omega} |p_k|_{1,s,\Omega} |p_l|_{1,\omega_i} + C \sum_j h^{1-2/s} |p_j|_{1,s,\Omega} |p_j|_{1,\omega_i} \\
&\le C \sum_l h^{2-4/s} |p_l|_{1,\omega_i} + C \sum_j h^{1-2/s} |p_j|_{1,\omega_i} \\
&\le C h^{1-2/s} \sum_l |p_l|_{1,\omega_i}.
\end{aligned}
\tag{3.107}
$$

It can be easily shown that

$$
\begin{aligned}
|R_i|_{0,\omega_i} &\le C \sum_{j,k} \|(p_j - \overline{p}_j^i)(p_k - \overline{p}_k^i)\|_{0,\omega_i} \\
&\le C \sum_{j,k} h^{2-4/s} |p_j|_{1,s,\omega_i} |p_k|_{1,s,\omega_i} h \\
&\le C h^{3-4/s} \sum_j |p_j|_{1,s,\omega_i}.
\end{aligned}
\tag{3.108}
$$

Following [16], we have

$$
\begin{aligned}
|G(p_1, ..., p_N) - c_{ij}\psi_{ij}|^2_{1,\Omega} &= \int_\Omega |\nabla (G - c_{ij}\phi_i^0 p_j)|^2 dx = \int_\Omega |\nabla(\phi_i^0 (G - c_{ij} p_j))|^2 dx \\
&\le C \int_\Omega |(G - c_{ij}p_j)\nabla \phi_i^0|^2 dx + C \int_\Omega |\phi_i^0 \nabla (G - c_{ij}p_j)|^2 dx \\
&\le \frac{1}{h^2} \sum_i \int_{\omega_i} |R_i|^2 dx + \sum_i \int_{\omega_i} |\nabla R_i|^2 dx \\
&\le C \sum_i \frac{1}{h^2} h^{6-8/s} \sum_j |p_j|^2_{1,s,\omega_i} + C h^{2-4/s} \sum_i \sum_j |p_j|^2_{1,\omega_i} \\
&\le C h^{4-8/s} + C h^{2-4/s},
\end{aligned}
\tag{3.109}
$$

where we have used the fact that $\sum_i \phi_i^0 = 1$ and C depends on the overlapping index of ω_i's. Consequently, we have the following error estimate

$$|p - p_h|_{1,\Omega} \leq C\delta + Ch^{1-2/s}. \tag{3.110}$$

Theorem 3.3.4. *Assume (3.105) and* $p_i \in W^{1,s}(\Omega)$, $s > 2$, $i = 1, ..., N$. *Then*

$$|p - p_h|_{1,\Omega} \leq C\delta + Ch^{1-2/s}.$$

3.3.2. Harmonic Coordinate System Methods

In this section, we introduce multiscale numerical methods based on harmonic coordinates [53]. This is an extension of the Galerkin FEM in [53] to mixed FEM. The idea is to find the global fields p_i ($i = 1, \cdots, d$, where $d = \dim(\Omega)$) so that p_i satisfy the following equation

$$\begin{aligned} div(k(x)\nabla p_i) = 0 \quad \text{in} \quad \Omega \\ p_i = x_i \quad \text{on} \quad \partial\Omega. \end{aligned} \tag{3.111}$$

Let $F = (p_1, \cdots, p_d)$, then one can show that $p \circ F^{-1} \in C^{1,\alpha}$. Let L_i be the standard linear base function associated to the vertex $(p_1(x_i), \cdots, p_d(x_i))$. We define the Galerkin finite element basis by

$$\psi_i = L_i \circ F.$$

Let p_h be the solution of (3.77) using the Galerkin finite element basis function ψ_i, one can obtain [53]

$$\|p - p_h\|_{1,\Omega} \leq Ch^{\alpha}\|f\|_{\infty,\Omega},$$

where α depends on Ω, $\frac{\lambda_{max}(k(x))}{\lambda_{min}(k(x))}$ and $essup_{x\in\Omega}(\frac{\lambda_{\max}(\nabla F^t k \nabla F)}{\lambda_{\max}(\nabla F^t k \nabla F)})$. This is the convergence rate of the Galerkin FEM in harmonic coordinates $(p_1, \cdots, p_d)$.

Next we will discuss the mixed FEM in the harmonic coordinate system. Let V_h^F

be the velocity finite element space in $(p_1, \cdots, p_d)$. Define the velocity basis function space in $x = (x_1, \cdots, x_d)$ coordinate system by the Piola transform [19], i.e.,

$$V_h^x = \{\frac{1}{J(F^{-1})} D(F^{-1}) v_h^F | v_h^F \in V_h^F\},$$

where $D(F^{-1})$ is the Jacobian matrix of F^{-1} and $J(F^{-1}) = |detD(F^{-1})|$. Let Q_h^F be the pressure basis function space in the harmonic $(p_1, \cdots, p_d)$ system. Define the pressure basis function space in $x = (x_1, \cdots, x_d)$ system by

$$Q_h^x = \{q_h^F(F(x)) | q_h^F \in Q_h^F\}.$$

Let $p^F = p \circ F^{-1}$, $k^F = \frac{1}{J(F)} D(F) k D(F)^t$ and $u^F = -k^F \nabla_F p^F$, where ∇_F is the gradient operator with respect to the variable $(p_1, \cdots, p_d)$. Then we can obtain the following lemma.

Lemma 3.3.5. *[57]*

$$u = -k \nabla p = \frac{1}{J(F^{-1})} D(F^{-1}) u^F.$$

By using Lemma 3.3.5 and change of variables, we have the following theorem provided Q_h^F is the piecewise constant space and $V_h^F = RT_0$.

Theorem 3.3.6. *Let $p_h \in Q_h^x$ be the numerical approximation of the pressure p and $u_h \in V_h^x$ be the numerical approximation of the velocity of $u = -k\nabla p$. Then*

$$\|p - p_h\|_{0,\Omega} + \|u - u_h\|_{H(div,\Omega)} \leq C(F)(\|\nabla_F p^F\|_{0,\Omega_F} + |u^F|_{1,\Omega_F} + |div_F u^F|_{1,\Omega_F})h,$$

where $C(F) = C \max\{(\sup_x J(F))^{1/2} \|D(F)^{-1}\|_{L^\infty(\Omega)}, (\sup_x J(F))^{1/2}, (\inf_x J(F))^{-1/2}\}$ and h is the mesh size in $(p_1, \cdots, p_d)$ system.

Proof. Let $\hat{x}$ be the point in coordinate of $\{p_1, \cdots, p_d\}$. We first notice

$$\begin{aligned}\|p - p_h\|_{0,\Omega} &= (\int_{\Omega_F} (p^F - p_h^F)^2 J(F^{-1}) d\hat{x})^{1/2} \\ &\leq (\sup_{\hat{x}\in\Omega_F})^{1/2} \|p^F - p_h^F\|_{0,\Omega_F} \\ &\leq C(\inf J(F))^{-1/2} |\nabla_F p^F|_{0,\Omega_F} h.\end{aligned} \tag{3.112}$$

Invoking Lemma 3.3.5, it follows

$$\begin{aligned}\|u - v_h\|_{0,\Omega} &= (\int_{\Omega_F} |\frac{1}{J(F^{-1})}||D(F^{-1})|^2 (u^F - v_h^F)^2 d\hat{x})^{1/2} \\ &\leq (\inf J(F^{-1}))^{-1/2} \|D(F^{-1})\|_{L^\infty(\Omega_F)} \|u^F - v_h^F\|_{0,\Omega_F} \\ &\leq C(\sup J(F))^{1/2} \|D(F)^{-1}\|_{L^\infty(\Omega)} |u^F|_{1,\Omega_F} h.\end{aligned} \tag{3.113}$$

Noting that $div(u - v_h) = \frac{1}{J(F^{-1})} div_F(u^F - v_h^F)$ and we have

$$\begin{aligned}\|div(u - v_h)\|_{0,\Omega} &= (\int_{\Omega_F} J(F)(div_F(u^F - v_h^F))^2 d\hat{x})^{1/2} \\ &\leq (\sup J(F))^{1/2} \|div_F(u^F - v_h^F)\|_{0,\Omega_F} \\ C &\leq (\sup J(F))^{1/2} |div_F u^F|_{1,\Omega_F} h.\end{aligned} \tag{3.114}$$

Combining (3.112), (3.113),(3.114) and the stability estimate [19], i.e.,

$$\|p - p_h\|_{0,\Omega} + \|u - u_h\|_{H(div,\Omega)} \leq C(\inf_{q_h\in Q_h^x} \|p - q_h\|_{0,\Omega} + \inf_{v_h\in V_h^x} \|u - v_h\|_{H(div,\Omega)}), \tag{3.115}$$

we complete the proof. □

If we use weighted norms, then we can describe the Theorem 3.3.6 as following.

Proposition 3.3.7. *Let $p_h \in Q_h^x$ be the numerical approximation of the pressure p and $u_h \in V_h^x$ be the numerical approximation of the velocity of $u = -k\nabla p$. If $\|D(F)^{-1}\|_{L^\infty(\Omega)} \leq C$, then*

$$\|p - p_h\|_{L^2_{J(F)}(\Omega)} + \|u - u_h\|_{H_{J(F)^{-1}}(div,\Omega)} \leq Ch(\|\nabla_F p^F\|_{0,\Omega_F} + |u^F|_{1,\Omega_F} + |div_F u^F|_{1,\Omega_F}),$$

where $\|.\|_{L^2_{J(F)}(\Omega)}$ *is the weighted* L^2 *norm with weight* $J(F)$ *and* $\|.\|_{H_{J(F)^{-1}}(div,\Omega)}$ *is the weighted* $H(div)$ *norm with weight* $J(F)^{-1}$.

Proof. By the proof of (3.112), it follows that

$$\|p - p_h\|_{L^2_{J(F)}(\Omega)} = \|p^F - p_h^F\|_{0,\Omega_F} \le Ch|\nabla_F p^F|_{0,\Omega_F}.$$

From the proof of (3.113), we have

$$\|u - v_h\|_{L^2_{J(F)^{-1}}(\Omega)} \le Ch|u^F|_{1,\Omega_F},$$

where we have used the assumption $\|D(F)^{-1}\|_{L^\infty(\Omega)} \le C$. Proof of (3.114) implies that

$$\|div(u - v_h)\|_{L^2_{J(F)^{-1}}(\Omega)} = \|div_F(u^F - v_h^F)\|_{0,\Omega_F} \le Ch|div_F u^F|_{1,\Omega_F}.$$

Using the stability estimate completes the proof. $\square$

Remark 3.3.2. *For a further rigorous proof, we need to show inf-sup condition for the mixed finite element method in the space* V_h^x *and* Q_h^x *and estimate the terms* $\|\nabla_F p^F\|_{0,\Omega_F}$, $\|u^F\|_{1,\Omega_F}$ *and* $\|div_F u^F\|_{1,\Omega_F}$ *in terms of source term* f*. One of the challenges for the questions is that the partial differential equation (3.76) becomes a non-divergence partial differential equation in the harmonic coordinate system* $(p_1, \cdots, p_d)$*. This problem is currently under investigation.*

3.3.3. Mixed MsFEM Using Multiple Global Information

In this section, we propose a mixed MsFEM using multiple limited global information and investigate its application to heterogeneous media. A rigorous analysis for this method is presented. We apply the global mixed MsFEM to two-phase flows with parameterized permeability and present numerical results in Section 3.4.3.

Global Mixed MsFEM . In this section, we study the following elliptic problem with heterogenous coefficients

$$
\begin{aligned}
-div(\lambda(x)k(x)\nabla p) = f(x) \quad \text{in} \quad \Omega \\
\lambda(x)k(x)\nabla p\cdot n = g \quad \text{on} \quad \partial\Omega,
\end{aligned}
\tag{3.116}
$$

where where $k(x)$ is a heterogeneous field and $\lambda(x)$ is a smooth field.

Let $u = \lambda(x)k(x)\nabla p$ be the velocity and $\Omega \subset R^2$ be convex. We introduce a quasi-uniform finite element partition τ_h of Ω and let K be a representative triangle (coarse), $h = \max_K diam(K)$. In the analysis, we will use the following assumption.

Assumption A1. There exist functions $u_1, \cdots, u_N$ and sufficiently smooth $A_1(x), \cdots, A_N(x)$ such that

$$
u(x) = A_i(x)u_i, \tag{3.117}
$$

where $u_i = k\nabla p_i$ and p_i solves $div(k(x)\nabla p_i) = 0$ in Ω with appropriate boundary conditions.

We note that summation convention is used throughout the dissertation. For our analysis, we assume $A_i(x) \in W^{1,\xi}(\Omega)$, and $u_i = k(x)\nabla p_i \in L^{\eta}(\Omega)$ for some ξ and η, $i = 1, ..., N$. We note that summation convention is used throughout the dissertation.

Remark 3.3.3. *As an example of two global fields in R^2, we use the results in [53]. Let $u_i = k(x)\nabla p_i$ $(i = 1, 2)$ be defined by the elliptic equation*

$$
\begin{aligned}
div(k(x)\nabla p_i) = 0 \quad in \quad \Omega \\
p_i = x_i \quad on \quad \partial\Omega,
\end{aligned}
\tag{3.118}
$$

where $x = (x_1, x_2)$. In the harmonic coordinate (p_1, p_2), $p(p_1, p_2) \in W^{2,s}$ $(s \geq 2)$ [53]. Consequently, $u = \lambda(x)k(x)\nabla p = \lambda\frac{\partial p}{\partial p_i}k\nabla p_i := A_i(x)u_i$, where $A_i(x) = \lambda\frac{\partial p}{\partial p_i} \in W^{1,s}$.

Let λ be a positive smooth function and k is a positive definite tensor so that $(\lambda k)^{-1}$ exists. The mixed formulation is to find $\{u, p\} \in H(div, \Omega) \times L^2(\Omega)/R$ such that $u \cdot n = g$ on $\partial\Omega$ and

$$
\begin{aligned}
((\lambda k)^{-1}u, v) + (divv, p) = 0 \quad \forall v \in H_0(div, \Omega) \\
-(divu, q) = (f, q) \quad \forall q \in L^2(\Omega)/R,
\end{aligned}
\tag{3.119}
$$

where $(\cdot, \cdot)$ is the usual L^2-inner product.

To numerically approximate the mixed problem (3.119), we construct the basis function for velocity,

$$
\begin{cases}
div(k(x)\nabla\phi_{ij}^K) & = \frac{1}{|K|} \quad \text{in} \quad K \\
k(x)\nabla\phi_{ij}^K \cdot n_{e_l} & = \delta_{jl}\dfrac{u_i \cdot n_{e_l}}{\int_{e_l} u_i \cdot n_{e_l} ds} \quad \text{on} \quad \partial K \\
\int_K \phi_{ij}^K dx & = 0,
\end{cases}
\tag{3.120}
$$

where $i = 1, ..., N$, $j = 1, 2, 3$, e_l is an edge of ∂K, and

$$\delta_{jj} = 1, \quad \delta_{jl} = 0 \ \text{ if } \ j \neq l.$$

Here e_l denotes an edge of the triangle and we omit the subscript e_l in n, if the integral is taken along the edge. Note that for each edge, we have N basis functions and we assume that u_1,..., u_N are linearly independent in order to guarantee that the basis functions are linearly independent. To avoid the possibility that $\int_{e_l} u_i \cdot nds$ is zero or unbounded, we make the following assumption for our analysis.

Assumption A2. There exist positive constants C such that for any u_i,

$$\int_{e_l} |u_i \cdot n| ds \leq Ch^{\beta_1} \quad and \quad \|\frac{u_i \cdot n}{\int_{e_l} u_i \cdot nds}\|_{L^r(e_l)} \leq Ch^{-\beta_2 + \frac{1}{r} - 1}$$

uniformly for all edges e_l, where $\beta_1 \leq 1$, $\beta_2 \geq 0$,and $r \geq 1$.

Remark 3.3.4. *The second part of Assumption A2 is to assure $|\int_{e_l} u_i \cdot nds|$ remains positive. It can be also written as $\|\frac{u_i \cdot n}{\int_{e_l} u_i \cdot nds} - \langle\frac{u_i \cdot n}{\int_{e_l} u_i \cdot nds}\rangle_{e_l}\|_{L^r(e_l)} \leq Ch^{-\beta_2 + \frac{1}{r} - 1}$, where $\langle\cdot\rangle = \frac{1}{|e_l|}\int_{e_l}(\cdot)ds$, which will be used to estimate the velocity basis function. If u_i*

are bounded in $L^\infty(e_l)$ *for all* e_l *and* $|\int_{e_l} u_i \cdot n ds|$ *remains positive uniformly for all* e_l*, then* $\beta_2 = 0$*. Note that* $\|\frac{u_i \cdot n}{\int_{e_l} u_i \cdot n ds} - \langle \frac{u_i \cdot n}{\int_{e_l} u_i \cdot n ds}\rangle_{e_l}\|_{L^r(e_l)} = 0$ *if* $u_i|_K$ *is* RT_0 *basis function or standard mixed MsFEM basis functions introduced in [22]. Finally, if* $|\int_{e_l} u_i \cdot n ds| \geq Ch^{\beta_1}$ *and* $\int_{e_l} |u_i \cdot n| ds \leq Ch^{\beta_1}$ *for all* e_l*, then we can conclude that* $\beta_2 = 0$ *for* $r = 1$ *in Assumption A2.*

We define $\psi_{ij}^K = k(x)\nabla\phi_{ij}^K$ and

$$V_h = \bigoplus_K \{\psi_{ij}^K\} \bigcap H(div, \Omega)$$

and

$$V_h^0 = \bigoplus_K \{\psi_{ij}^K\} \bigcap H_0(div, \Omega).$$

Let $Q_h = \oplus_K P_0(K) \subset L^2(\Omega)/R$, i.e., piecewise constants, be the basis function for the pressure. We define

$$g_{0,h} = \sum_{e \in \{\partial K \bigcap \partial\Omega, K \in \tau_h\}} (\int_e g ds)\psi_{i,e}$$

for some fixed $i \in \{1, 2, \cdots, N\}$, where $\psi_{i,e}$ is the corresponding multiscale basis function to the edge e. Let $g_h = g_{0,h} \cdot n$ on $\partial\Omega$. The numerical mixed formulation is to find $\{u_h, p_h\} \in V_h \times Q_h$ such that $u_h \cdot n = g_h$ on $\partial\Omega$ and

$$\begin{aligned} ((\lambda k)^{-1} u_h, v_h) + (div v_h, p_h) = 0 \quad \forall v \in V_h^0 \\ -(div u_h, q_h) = (f, q_h) \quad \forall q_h \in Q_h. \end{aligned} \tag{3.121}$$

First, we note the following result.

Lemma 3.3.8.

$$u_i|_K \in span\{\psi_{ij}^K\}, \quad i = 1, .., N; \ j = 1, 2, 3.$$

Proof. First, we prove the lemma for u_1. For this proof, we want to find constants

β_{ij}^K's such that $\beta_{ij}^K \psi_{ij}^K = u_1$. That is

$$\begin{aligned} \beta_{ij}^K k(x)\nabla\phi_{ij}^K \cdot n_{e_l}^K &= \beta_{ij}^K \delta_{jl} \frac{u_i \cdot n_{e_l}^K}{\int_{e_l} u_i \cdot nds} = u_1 \cdot n_{e_l}^K \\ \beta_{ij}^K div(k(x)\nabla\phi_{ij}^K) &= \frac{1}{|K|}\beta_{ij}^K = 0. \end{aligned} \tag{3.122}$$

Noticing that $u_i = k(x)\nabla p_i$ and $div(k(x)\nabla p_i) = 0$, we have $p_i = \beta_{ij}^K \phi_{ij}^K + C$ for some constant C because p_i and $\beta_{ij}^K \phi_{ij}^K$ satisfy the same elliptic equation with Neumann boundary condition as p_i. Then we have $u_i = \beta_{ij}^K \psi_{ij}^K$. The first equation in (3.122) implies that we can take $\beta_{1j}^K = \int_{e_j} u_1 \cdot nds$ and $\beta_{ij}^K = 0$ for $i \neq 1$. Consequently,

$$\sum_{i,j} \beta_{ij}^K = \sum_j \int_{e_j} u_1 \cdot nds = \int_K divu_1 dx = 0,$$

which is the first equation in (3.122). We can obtain similar results for other u_i's $(i = 2, ..., N)$. □

Following our assumption, let

$$X = \{u | u = a_i(x)u_i\}$$

be a subspace of $H(div, \Omega)$. For our analysis, we require that the integrals $\displaystyle\int_{e_j} a_i(x)u_i \cdot nds$ are well defined for each i. This is also needed in our computations since $\displaystyle\int_{e_j} a_i(x)u_i \cdot nds$ determines the fluxes along the edges in two-phase flow simulations. One way to achieve this is to assume, as we did earlier, that $a_i(x) \in W^{1,\xi}(\Omega)$, $u_i \in L^\eta(\Omega)$, $\frac{1}{2} = \frac{1}{\xi} + \frac{1}{\eta}$. Because $a_i(x) \in W^{1,\xi}(\Omega)$ and $u_i \in L^\eta(\Omega)$ $(\frac{1}{2} = \frac{1}{\xi} + \frac{1}{\eta})$, Hölder inequality implies that $(\nabla a_i)u_i \in L^2(\Omega)$. Noticing that $divu_i = 0$, we have $div(a_i(x)u_i) \in L^2(\Omega)$ immediately. Invoking standard Sobolev embedding theorems 2.2.4 (also c.f. [6]), we get $a_i u_i \in L^\eta(\Omega)$ because $W^{1,\xi} \hookrightarrow L^\infty$. The integrals $\displaystyle\int_{e_j} a_i(x)u_i \cdot nds$ are well defined by the fact that $a_i u_i \in L^\eta(\Omega)$ $(\eta > 2)$ and $div(a_i(x)u_i) \in L^2(\Omega)$ (see page 125 of [19]). This can be proved by using Green's formula and standard Sobolev embedding theorems.

Remark 3.3.5. *For all* $\int_{e_j} a_i(x)u_i \cdot n ds$ *to be well defined, one sufficient condition is that* $a_i u_i \in [L^p(K)]^d$ *and* $div(a_i(x)u_i) \in L^s(K)$ *for* $p > 2$, $s \geq q$, $\frac{1}{q} = \frac{1}{p} + \frac{1}{d}$ *[34]. We give a brief proof for the case* $a_i(x) \in W^{1,\xi}(K)$ *and* $u_i \in L^\eta(K)$, $\frac{1}{2} = \frac{1}{\xi} + \frac{1}{\eta}$. *We have previously proved that* $a_i u_i \in L^\eta(K)$ *and* $div(a_i u_i) \in L^2(K)$. *Let* $\frac{1}{\eta} + \frac{1}{\eta'} = 1$ $(\eta > 2)$ *and* $e \subset \partial K$. *Then exists a function* $w \in W^{1,\eta'}(K)$ *such that* $w|_e = 1$ *and* $w|_{\partial K \setminus e} = 0$ *(see page 78 in [34]). Applying Green's formula, we have*

$$\int_K (a_i u_i) \cdot \nabla w dx + \int_K (div(a_i u_i)) w dx = \int_{\partial K} \gamma(w)(a_i u_i) \cdot n ds = \int_e (a_i u_i) \cdot n ds, \quad (3.123)$$

where $\gamma(w)$ *is the trace of* w *along boundary of* K. *Since* $\frac{1}{\eta} + \frac{1}{\eta'} = 1$, *Hölder inequality implies that the term* $\int_K a_i u_i \cdot \nabla w dx$ *is integrable. Since* $w \in W^{1,\eta'}(K)$, *Sobolev embedding theorem 2.2.4 imply that* $W^{1,\eta'}(K) \hookrightarrow L^2(K)$. *By combining* $w \in L^2(K)$ *with* $div(a_i u_i) \in L^2(K)$, *we obtain that the term* $\int_K (div(a_i u_i)) w dx$ *is integrable. Consequently, (3.123) implies that* $\int_e (a_i u_i) \cdot n ds$ *is meaningful. We note that no summation convention is used here.*

Remark 3.3.6. *The integral* $\int_e a_i u_i \cdot n ds$ *can be controlled by* $\|u_i\|_{L^\eta(K)}$ *for any fixed* u_i. *In fact, let* w *in Remark 3.3.5 such that* $\|w\|_{W^{1,\eta'}(K)} = 1$. *By (3.123), we have*

$$\begin{aligned}
|\int_e a_i u_i \cdot n ds| &\leq \|a_i u_i\|_{L^\eta(K)} \|\nabla w\|_{L^{\eta'}(K)} + \|div(a_i u_i)\|_{L^2(K)} \|w\|_{L^2(K)} \\
&\leq \|a_i u_i\|_{L^\eta(K)} \|\nabla w\|_{L^{\eta'}(K)} + \|div(a_i u_i)\|_{L^2(K)} \|w\|_{W^{1,\eta'}(K)} \\
&\leq \|a_i u_i\|_{L^\eta(K)} + \|(\nabla a_i) u_i\|_{L^2(K)} \\
&\leq \|a_i\|_{L^\infty(K)} \|u_i\|_{L^\eta(K)} + \|\nabla a_i\|_{L^\xi(K)} \|u_i\|_{L^\eta(K)} \\
&\leq C \|u_i\|_{L^\eta(K)},
\end{aligned} \quad (3.124)$$

where Sobolev embedding theorem has been used.

Thus, more precise definition of space X is

$$X = \{u | u = a_i(x)u_i : a_i \in W^{1,\xi}(\Omega),\ u_i \in L^{\eta}(\Omega)\}, \tag{3.125}$$

where $\frac{1}{\xi} + \frac{1}{\eta} = \frac{1}{2}$. We define an interpolation operator $\Pi_h : X \longrightarrow V_h$ such that in each element K, for any $v = a_i(x)u_i \in X$

$$\Pi_h|_K(a_i(x)u_i) = a_{ij}^K \psi_{ij}^K,$$

where $a_{ij}^K = \displaystyle\int_{e_j} a_i(x)u_i \cdot n ds$. Utilizing the definition of Π_h, we obtain the following lemma.

Lemma 3.3.9. *Let Π_h be defined as above. Then $\forall v = a_i u_i \in X$ (cf. (3.125)) and $q_h \in Q_h$,*

(1) $\int_\Omega div(v - \Pi_h v) q_h dx = 0$;

(2) $\|\Pi_h v\|_{H(div,\Omega)} \leq C\|v\|_{X,\Omega}$, if $\beta_1 \geq 2\beta_2$,

where $\|v\|_{X,\Omega} := \|div(v)\|_{0,\Omega} + \sum_{i=1}^N \|a_i\|_{1,\Omega}$ and C only depends on N, the constants in Assumption A2 and the pre-computed global fields u_i.

Proof. For a better description of the proof, we use summation notation in this proof instead of summation convention. (1). For this, we only need to show that on each e_l, $\int_{e_l}(v - \Pi_h v) \cdot n[q_h]_{e_l} ds = 0$, where $[q_h]_{e_l}$ is the jump of q_h between two sides of e_l. It is sufficient to show $\int_{e_l}(v - \Pi_h v) \cdot nds = 0$ since $q_h \in Q_h$. In fact,

$$\begin{aligned}
\sum_i \int_{e_l} (a_i(x)u_i \cdot n - \sum_j a_{ij}^K \psi_{ij} \cdot n) ds &= \sum_i \int_{e_l} (a_i(x)u_i \cdot n - \sum_j a_{ij}^K \delta_{jl} \frac{u_i \cdot n}{\int_{e_l} u_i \cdot n}) ds \\
&= \sum_i \int_{e_l} (a_i(x)u_i \cdot n - a_{il}^K \frac{u_i \cdot n}{\int_{e_l} u_i \cdot n}) ds \\
&= \sum_i (\int_{e_l} a_i(x)u_i \cdot nds - \int_{e_l} a_i(x)u_i \cdot nds) \\
&= 0.
\end{aligned} \tag{3.126}$$

(2) In each element K, for any $v = \sum_i a_i(x)u_i$, we have

$$\begin{aligned}
\|div\Pi_h v\|_{0,K}^2 &= \|\sum_{i,j}\int_{e_j} a_i(x)u_i \cdot nds\frac{1}{|K|}\|_{0,K}^2 \\
&= \|\sum_i \langle div(a_i(x)u_i)\rangle_K\|_{0,K}^2 \\
&\leq C\|\sum_i div(a_i(x)u_i)\|_{0,K}^2 \\
&= C\|divv\|_{0,K}^2,
\end{aligned}$$

where $\langle\cdot\rangle_K$ denotes the volume average over K and we have used divergence theorem in the second step and Jensen's inequality in the third step. After summing all over K, we get

$$\|div\Pi_h v\|_{0,\Omega} \leq C\|divv\|_{0,\Omega} \tag{3.127}$$

where C only depends on N.

Let $v = \sum_i a_i u_i$, $\bar{a}_i^K = \frac{1}{|K|}\int_K a_i dx$ and $\frac{1}{r} + \frac{1}{r'} = 1 = \frac{1}{\eta} + \frac{1}{\eta'}$, then

$$\begin{aligned}
&\|\Pi_h v\|_{0,K}^2 \\
&= \|\sum_{i,j}(\int_{e_j}(a_i - \bar{a}_i^K)u_i \cdot nds\psi_{ij}^K) + \sum_{i,j}(\bar{a}_i^K\int_{e_j} u_i \cdot nds\psi_{ij}^K)\|_{0,K}^2 \\
&\leq 2\|\sum_{i,j}(\int_{e_j}(a_i - \bar{a}_i^K)u_i \cdot nds\psi_{ij}^K)\|_{0,K}^2 + 2\|\sum_i(\bar{a}_i^K u_i)\|_{0,K}^2 \\
&\leq 2\sum_{i,j}|\int_{e_j}(a_i - \bar{a}_i^K)u_i \cdot nds)|^2\sum_{i,j}\|\psi_{ij}^K\|_{0,K}^2 + 4\|\sum_i(\bar{a}_i^K u_i) - v\|_{0,K}^2 + 4\|v\|_{0,K}^2 \qquad (3.128) \\
&\leq C\sum_{i,j}\|a_i - \bar{a}_i^K\|_{L^{r'}(e_j)}^2\|u_i \cdot n\|_{L^r(e_j)}^2\sum_{i,j}\|\psi_{ij}^K\|_{0,K}^2 + 4\|\sum_i(\bar{a}_i^K - a_i)u_i\|_{0,K}^2 + 4\|v\|_{0,K}^2 \\
&\leq Ch^{\frac{2}{r'}}\|\nabla a_i\|_{0,K}^2 h^{2\beta_1 - 2\beta_2 + \frac{2}{r} - 2}h^{-2\beta_2} + C\sum_i\|a_i - \bar{a}_i^K\|_{L^{\frac{2\eta}{\eta-2}}(K)}^2\|u_i\|_{L^\eta(K)}^2 + 4\|v\|_{0,K}^2 \\
&\leq Ch^{2\beta_1 - 4\beta_2}\sum_i\|\nabla a_i\|_{0,K}^2 + C\sum_i\|\nabla a_i\|_{L^{\eta'}(K)}^2\|u_i\|_{L^\eta(K)}^2 + 4\|v\|_{0,K}^2,
\end{aligned}$$

where we have used Lemma 3.3.8 in the second step, Schwarz inequality in the third and forth step, *Assumption A2* and (3.140) along with Sobolev embedding inequality (by rescaling) in the fifth step and Poincaré-Friedrichs inequality (by rescaling) in the sixth step. Making summation all over for K in the above, we have

$$\begin{aligned}
\|\Pi_h v\|_{0,\Omega}^2 &\le Ch^{2\beta_1-4\beta_2}\sum_i \|\nabla a_i\|_{0,\Omega}^2 + C\sum_i (\|\nabla a_i\|_{L^{\eta'}(\Omega)}^2 \|u_i\|_{L^\eta(\Omega)}^2) + 4\|v\|_{0,\Omega}^2 \\
&\le C(h^{2\beta_1-4\beta_2} + \max_i \|u_i\|_{L^\eta(\Omega)}^2)\|\nabla a_i\|_{0,\Omega}^2 + 4\|v\|_{0,\Omega}^2 \\
&\le C(h^{2\beta_1-4\beta_2} + \max_i \|u_i\|_{L^\eta(\Omega)}^2)\|\nabla a_i\|_{0,\Omega}^2 + C\sum_i (\|a_i\|_{L^{\frac{2\eta}{\eta-2}}(\Omega)}^2 \|u_i\|_{L^\eta(\Omega)}^2) \\
&\le C(h^{2\beta_1-4\beta_2} + \max_i \|u_i\|_{L^\eta(\Omega)}^2)\|a_i\|_{1,\Omega}^2,
\end{aligned} \tag{3.129}$$

where we have used Sobolev embedding inequality in the last step. Combining (3.127) and (3.129), we conclude

$$\|\Pi_h v\|_{H(div,\Omega)} \le C\|v\|_{X,\Omega},$$

where the C is independent of h and v, and only depends on N, constants in *Assumption A2*, $\|u_i\|_{L^\eta(\Omega)}$ and Sobolev embedding constant. □

Remark 3.3.7. *If $u_i \in L^\infty(\Omega)$, then $\beta_1 = 1$, $\beta_2 = 0$ and the proof of Lemma 3.3.9 implies that $\|\Pi_h v\|_{H(div,\Omega)} \le C(\max_i \|u_i\|_{L^\infty(\Omega)})\sum_i \|a_i\|_{1,\Omega}$.*

We can get a stability of Π_h different from the one in Lemma 3.3.9.

Proposition 3.3.10. *For $v = a_i u_i$, where $a_i \in W^{1,\xi}(\Omega)$ and $u_i \in L^\eta(\Omega)$ ($\frac{1}{2} = \frac{1}{\xi} + \frac{1}{\eta}$), then*

$$\|\Pi_h v\|_{H(div,\Omega)} \le C\sum_i \|a_i\|_{W^{1,\xi}(\Omega)},$$

if $\alpha + \beta_1 - \beta_2 - 1 \ge 0$, where C only depends on N, the constants in Assumption A2 and the pre-computed global fields u_i.

Proof. For a better description of the proof, we use summation notation in this proof instead of summation convention. By (3.127), we get

$$\begin{aligned}
\|div\Pi_h v\|_{0,\Omega} &\le C\|divv\|_{0,\Omega} \\
&= C\|\sum_i (\nabla a_i)u_i\|_{0,\Omega} \\
&\le C\sum_i \|\nabla a_i u_i\|_{0,\Omega} \\
&\le C\sum_i \|\nabla a_i\|_{L^\xi(\Omega)}\|u_i\|_{L^\eta(\Omega)} \\
&\le C\max_i\{\|u_i\|_{L^\eta(\Omega)}\}\sum_i \|\nabla a_i\|_{L^\xi(\Omega)},
\end{aligned} \tag{3.130}$$

where the C only depends on N. If we use the estimates (3.141) - (3.142), we find that

$$\|\Pi_h v - v\|_{0,\Omega} \le Ch^{\alpha+\beta_1-\beta_2-1}\sum_i \|a_i\|_{C^\alpha(\Omega)} \le Ch^{\alpha+\beta_1-\beta_2-1}\sum_i \|a_i\|_{W^{1,\xi}(\Omega)},$$

where we have used the Sobolev imbedding inequality in the second step and the C only depends on N, constants in *Assumption A2* and Sobolev imbedding constant. Consequently,

$$\begin{aligned}
\|\Pi_h v\|_{0,\Omega} &\le Ch^{\alpha+\beta_1-\beta_2-1}\sum_i \|a_i\|_{W^{1,\xi}(\Omega)} + \|v\|_{0,\Omega} \\
&\le Ch^{\alpha+\beta_1-\beta_2-1}\sum_i \|a_i\|_{W^{1,\xi}(\Omega)} + \sum_i \|a_i u_i\|_{0,\Omega} \\
&\le Ch^{\alpha+\beta_1-\beta_2-1}\sum_i \|a_i\|_{W^{1,\xi}(\Omega)} + \sum_i \|a_i\|_{L^\xi(\Omega)}\|u_i\|_{L^\eta(\Omega)} \\
&\le Ch^{\alpha+\beta_1-\beta_2-1}\sum_i \|a_i\|_{W^{1,\xi}(\Omega)} + \max_i \|u_i\|_{L^\eta(\Omega)}\sum_i \|a_i\|_{L^\xi(\Omega)} \\
&\le C(h^{\alpha+\beta_1-\beta_2-1}+1)\sum_i \|a_i\|_{W^{1,\xi}(\Omega)},
\end{aligned} \tag{3.131}$$

where the C only depends on N, constants in *Assumption A2* and Sobolev imbedding constant. Combining (3.130) and (3.131), we conclude

$$\|\Pi_h v\|_{div,\Omega} \le C \sum_i \|a_i\|_{W^{1,\xi}(\Omega)},$$

where the C is independent of h and v, and only depends on N, constants in *Assumption A2*, $\|u_i\|_{L^\eta(\Omega)}$ and Sobolev imbedding constants. □

Remark 3.3.8. *We note that $\|v\|_{X,\Omega}$ may not be a norm in general because $v = a_i u_i = 0$ may not imply that a_i are zero (this does not impact the derivation of discrete inf-sup condition). In the problem setting considered in this paper, one can assume that $\|v\|_{X,\Omega}$ is a norm. Indeed, a_i are coarse-scale functions, while u_i are fine-scale functions. Thus, in each coarse grid block, the linear combination $a_i u_i$ zero will imply that a_i are zero unless u_i are also coarse-scale functions. In the latter case, one can use standard mixed finite element basis functions. If $N = d$ (d being the dimension of the space), $\|v\|_{X,\Omega}$ is a norm when u_i are linearly independent. In the discrete setting, a_i are vectors defined on the coarse grid, while u_i are defined on the fine grid. If $a_i u_i$ is zero, this implies that the vectors u_i are linearly dependent, and thus, the basis functions are linearly dependent. One can also attempt to prove inf-sup condition using fine-scale discrete setting (cf. [47]) by defining a discrete norm in X and showing a lemma analogous to Lemma 3.3.9 in the discrete setting.*

Lemma 3.3.9 and continuous inf-sup condition imply the discrete inf-sup condition (see page 58 of [19]). In the paper, we assume that continuous inf-sup condition holds. For a special case $N = d$, the continuous inf-sup condition follows from [53]. We briefly highlight the main idea of the proof. The goal is to find v, such that $div(v) = q$, $v = a_i u_i$ and $\|v\|_{X,\Omega} \le C\|q\|_{0,\Omega}$. Following [53], one can consider a coordinate transformation from $(x_1, ..., x_d)$ to $(z_1, ..., z_d)$, where $div(u_i) = 0$, $u_i = k\nabla z_i$ in Ω and $z_i = x_i$ on $\partial\Omega$. Defining $v = k\nabla\phi = k\frac{\partial\phi}{\partial z_i}\nabla z_i = \frac{\partial\phi}{\partial z_i}u_i$ and $a_i = \frac{\partial\phi}{\partial z_i}$, one can show that $\|v\|_{X,\Omega} \le C\|q\|_{0,\Omega}$. The latter follows from $\|\phi\|_{W^{2,2}(\Omega)} \le C\|q\|_{0,\Omega}$ (e.g., [53]) which holds in $(z_1, ..., z_d)$ coordinate system. One can also prove the continuous inf-sup condition by solving $u_i \nabla a_i = q$ for a_i along the streamlines of u_i (streamlines are defined as level sets of Ψ, $curl(\Psi_i) = u_i$).

Assuming continuous inf-sup condition, we have that for any $q_h \in Q_h$, there exists a constant C such that

$$\sup_{v_h \in V_h} \frac{\int_\Omega div v_h q_h dx}{\|v_h\|_{H(div,\Omega)}} \ge C\|q_h\|_{0,\Omega}. \tag{3.132}$$

Because of the inf-sup condition (3.132), we have the following optimal approximation (see [19, 22]).

Lemma 3.3.11. *Let $\{u, p\}$ and $\{u_h, p_h\}$ be the solution of (3.119) and (3.121) respectively. Then*

$$\|u - u_h\|_{H(div,\Omega)} + \|p - p_h\|_{0,\Omega} \le C\{\inf_{v_h \in V_h, v_h - g_{0,h} \in V_h^0} \|u - v_h\|_{H(div,\Omega)} + \inf_{q_h \in Q_h} \|p - q_h\|_{0,\Omega}\}. \tag{3.133}$$

Next, we formulate our main result.

Theorem 3.3.12. *Let $\{u, p\}$ and $\{u_h, p_h\}$ be the solution of (3.119) and (3.121) respectively. If $\alpha + \beta_1 - \beta_2 - 1 > 0$, we have*

$$\|u - u_h\|_{H(div,\Omega)} + \|p - p_h\|_{0,\Omega} \le Ch^{\min\{\alpha+\beta_1-\beta_2-1,1\}},$$

where $\alpha = 1 - \frac{2}{\xi}$, ξ and A_i are defined in Assumption A1, and β_i $(i = 1, 2)$ are defined in Assumption A2. Here C is independent of h and depends on N, the constants in Assumption A2, $\|A_i\|_{W^{1,\xi}(\Omega)}$ $(i = 1, .., N)$ and $\|f\|_{1,\Omega}$.

Proof. For a better description of the proof, we use summation notation in this proof instead of summation convention. For the proof, we need to choose a proper v_h and a proper q_h such that the right hand side of (3.133) is small.

The second term on the right hand in (3.133) can be easily estimated. In fact, with the choice $q_h|_K = \langle p \rangle_K$, i.e., the average of p in K, we have

$$\inf_{q_h \in Q_h} \|p - p_h\|_{0,\Omega} \le Ch|p|_{1,\Omega}.$$

Next we try to find a $v_h \in V_h$, say $v_h|_K = \sum_{i,j} c_{ij}^K \psi_{ij}^K$, and estimate the first term on the right hand in (3.133). Invoking Lemma 3.3.8 and its proof, it follows that in each K

$$\begin{aligned} u - v_h &= \sum_i A_i(x) u_i - \sum_{i,j} c_{ij}^K \psi_{ij} \\ &= \sum_i (A_i(x) \sum_j \beta_{ij}^K \psi_{ij}^K) - \sum_{i,j} c_{ij}^K \psi_{ij}^K \\ &= \sum_{i,j} (A_i(x) \beta_{ij}^K - c_{ij}^K) \psi_{ij}^K, \end{aligned} \tag{3.134}$$

where $\beta_{ij}^K = \int_{e_j} u_i \cdot nds$. Set $c_{ij}^K = A_{ij}^K = \int_{e_j} A_i(x) u_i \cdot nds.$

Since $\int_K \sum_i div(A_i(x) u_i) dx = f$, we get by divergence theorem

$$\int_{\partial K} \sum_i A_i(x) u_i \cdot nds = f.$$

This gives rise to

$$\begin{aligned} \|div(u - \sum_{i,j} c_{ij}^K \psi_{ij}^K)\|_{0,K} &= \|f - \sum_{i,j} c_{ij}^K \frac{1}{|K|}\|_{0,K} \\ &= \|f - \sum_{i,j} \int_{e_j} A_i(x) u_i \cdot nds \frac{1}{|K|}\|_{0,K} \\ &= \|f - \langle f \rangle_K\|_{0,K} \\ &\le Ch|f|_{1,K}. \end{aligned} \tag{3.135}$$

After making summation over all K for (3.135), we have

$$\|div(u - v_h)\|_{0,\Omega} \leq Ch|f|_{1,\Omega}. \tag{3.136}$$

Next we will estimate $\|u - \sum_{i,j} c_{ij}^K \psi_{ij}^K\|_{0,K}$. Because $A_i(x) \in W^{1,\xi}(\Omega)$, by using Sobolev embedding theorem and Taylor expansion (or definition of C^α) we have

$$|A_i(x)|_{e_j} - \bar{A}_i^j| \leq Ch^\alpha \|A_i\|_{C^\alpha(\Omega)},$$

where $\bar{A}_i^j$ is the average $A_i(x)$ along e_j and $\alpha = 1 - \frac{2}{\xi}$. So

$$\begin{aligned} |A_{ij}^K - \bar{A}_i^j \beta_{ij}^K| &= |\int_{e_j} A_i u_i \cdot nds - \bar{A}_i^j \int_{e_j} u_i \cdot nds| \\ &= |\int_{e_j} (A_i - \bar{A}_i^j) u_i \cdot nds| \\ &\leq Ch^{\alpha+\beta_1} \|A_i\|_{C^\alpha(\Omega)}, \end{aligned} \tag{3.137}$$

where we have used the *Assumption A2*.

Next, we present an estimate for $\|\psi_{ij}^K\|_{0,K}$. For this reason, we introduce lowest Raviart-Thomas basis functions R_j^K for velocity. We know that $divR_j^K = \frac{1}{|K|}$ and $R_j^K \cdot n = \delta_{jl} \frac{1}{|e_j|}$ [19]. We multiply (3.120) by a test function w we have

$$\begin{aligned} \int_K k\nabla\phi_{ij}^K \nabla w dx &= -\int_K wdiv(k\nabla\phi_{ij}^K)dx + \int_{\partial K} (k\nabla\phi_{ij}^K \cdot n) wds \\ &= -\int_K wdivR_j^K dx + \int_{\partial K} (k\nabla\phi_{ij}^K \cdot n) wds \\ &= \int_K (\nabla w) R_j^K dx + \int_{\partial K} (k\nabla\phi_{ij}^K \cdot n - R_j^K \cdot n) wds \\ &= \int_K (\nabla w) R_j^K dx + \int_{\partial K} \delta_{jl} \left(\frac{u_i \cdot n}{\int_{e_l} u_i \cdot nds} - \langle \frac{u_i \cdot n}{\int_{e_l} u_i \cdot nds} \rangle_{e_l}\right) wds, \end{aligned} \tag{3.138}$$

where we have used that $\langle \frac{u_i \cdot n}{\int_{e_j} u_i \cdot nds} \rangle_{e_j} = R_j^K \cdot n_{e_j} = \frac{1}{|e_j|}$. If we set $w = \phi_{ij}^K$, then it follows that

$$\begin{aligned} C\|\nabla\phi_{ij}^K\|_{0,K}^2 &\le \|\nabla\phi_{ij}^K\|_{0,K}\|R_j^K\|_{0,K} + \|\frac{u_i\cdot n}{\int_{e_j} u_i\cdot nds} - \langle\frac{u_i\cdot n}{\int_{e_j} u_i\cdot nds}\rangle_{e_j}\|_{L^r(e_j)}\|\phi_{ij}^K\|_{L^{r'}(\partial K)} \\ &\le C\|\nabla\phi_{ij}^K\|_{0,K} + Ch^{-\beta_2+\frac{1}{r}-1}\|\phi_{ij}^K\|_{L^{r'}(\partial K)} \\ &\le C\|\nabla\phi_{ij}^K\|_{0,K} + Ch^{-\beta_2+\frac{1}{r}-1}(h^{-1+\frac{1}{r'}}\|\phi_{ij}^K\|_{0,K} + h^{\frac{1}{r'}}\|\nabla\phi_{ij}^K\|_{0,K}) \\ &\le C\|\nabla\phi_{ij}^K\|_{0,K} + Ch^{-\beta_2+\frac{1}{r}-1}h^{\frac{1}{r'}}\|\nabla\phi_{ij}^K\|_{0,K} \\ &\le C\|\nabla\phi_{ij}^K\|_{0,K} + Ch^{-\beta_2}\|\nabla\phi_{ij}^K\|_{0,K}, \end{aligned} \tag{3.139}$$

where r' satisfies $\frac{1}{r}+\frac{1}{r'}=1$ (r is defined in *Assumption A2*), and we have used *Assumption A2* and $\|R_j^K\|_{0,K}\le C$ [19] in the second step, the trace inequality (by rescaling) in the third step and $\langle\phi_{ij}^K\rangle_K=0$ along with Poincaré-Friedrichs inequality (by rescaling) in the forth step. Consequently, we have

$$\|\psi_{ij}^K\|_{0,K}\le C(1+h^{-\beta_2}), \tag{3.140}$$

where C only depends on *AssumptionA2* and the constants in trace inequality and Poincaré inequality in a fixed reference domain. Combining (3.137) and (3.140), it follows immediately

$$\begin{aligned} \|u-v_h\|_{0,K} &= \|\sum_{i,j}(A_i(x)\beta_{ij}^K - A_{ij}^K)\psi_{ij}^K\|_{0,K} \\ &\le \|\sum_{i,j}(A_i(x)-\bar{A}_i^j)\beta_{ij}^K\psi_{ij}^K\|_{0,K} + \|\sum_{i,j}(\bar{A}_i^j\beta_{ij}^K - A_{ij}^K)\psi_{ij}^K\|_{0,K} \\ &\le \|\sum_{i,j}|A_i(x)-\bar{A}_i^j|\beta_{ij}^K\psi_{ij}^K\|_{0,K} + \|\sum_{i,j}|\bar{A}_i^j\beta_{ij}^K - A_{ij}^K|\psi_{ij}^K\|_{0,K} \\ &\le Ch^{\alpha+\beta_1}(\sum_i\|A_i\|_{C^\alpha(\Omega)})\sum_{i,j}\|\psi_{ij}^K\|_{0,K} \\ &\le Ch^{\alpha+\beta_1-\beta_2}(\sum_i\|A_i\|_{C^\alpha(\Omega)}), \end{aligned} \tag{3.141}$$

where we have used *Assumption A2* and the C depends on N and the constants in

Assumption A2. After making summation over all K for (3.141), we have

$$\begin{aligned}
\|u - v_h\|_{0,\Omega}^2 &\leq \sum_K \|u - v_h\|_{0,K}^2 \\
&\leq C(\sum_i \|A_i\|_{C^\alpha(\Omega)})^2 \sum_K h^{2(\alpha+\beta_1-\beta_2)} \\
&\leq C(\sum_i \|A_i\|_{C^\alpha(\Omega)})^2 \frac{1}{h^2} h^{2(\alpha+\beta_1-\beta_2)} \\
&= C(\sum_i \|A_i\|_{C^\alpha(\Omega)})^2 h^{2(\alpha+\beta_1-\beta_2-1)}.
\end{aligned}$$

Consequently,

$$\|u - v_h\|_{0,\Omega} \leq C(\sum_i \|A_i\|_{C^\alpha(\Omega)}) h^{\alpha+\beta_1-\beta_2-1}. \tag{3.142}$$

According to (3.133), for those K, $\partial K \cap \partial\Omega$, we will adjust proper c_{ij}^K such that $\sum_{i,j} c_{ij}^K \psi_{i,j}^K - g_{0,h} \in V_h^0$, but this will not affect our convergence rate. Therefore, invoking Lemma 3.3.11, (3.136) (3.142) and Sobolev embedding theorem from $W^{1,\xi}$ into C^α, Theorem 3.3.12 follows.

$\square$

From the proof of Theorem 3.3.12, one can easily get the following result, which does not assume inf-sup condition by lemma 5.3 in [3].

Corollary 3.3.13. *Let u and u_h be the velocity in (3.119) and (3.121) respectively, then we have*

$$\|u - u_h\|_{0,\Omega} \leq C(\sum_i \|A_i\|_{C^\alpha(\Omega)}) h^{\alpha+\beta_1-\beta_2-1}.$$

Remark 3.3.9. *If $A_i(x) \in C^1(\Omega)$ in Assumption A1 and u_i are defined such that $\beta_1 = 1$ and $\beta_2 = 0$ (e.g., u_i are bounded in $L^\infty(e_l)$ for all e_l), then Theorem 3.3.12 implies that*

$$\|u - u_h\|_{H(div,\Omega)} + \|p - p_h\|_{0,\Omega} \leq Ch.$$

Remark 3.3.10. *Mixed multiscale finite element method introduced in [22] is related RT_0 mixed finite element method in terms of degrees of freedom on the edges of elements. The proposed mixed multiscale method for $N = 2$ has similar relation to BDM_1 element (see [19] for the description of BDM_1). We note that the local mixed multiscale finite element methods suffer from a resonance error and a typical convergence rate for periodic coefficients is*

$$\|u_\epsilon - u_h\|_{H(div,\Omega)} + \|p_\epsilon - p_h\|_{0,\Omega} \leq C(h + \left(\frac{\epsilon}{h}\right)^\gamma),$$

where $\gamma = 1/2$ for mixed multiscale method introduced in [22]. In our global mixed MsFEM, the boundary condition for velocity basis is heterogenous and Theorem 3.3.12 implies that stability is independent of the small scale and the resonance error is removed.

Remark 3.3.11. *One can relax the main assumption used in the paper and assume that*

$$\|u(x) - A_i(x)u_i(x)\|_{H(div,\Omega)} \leq C\delta.$$

If $i = 1$, the proof can be found in [4]. For the general case, i.e., $i > 1$, we only need to verify the inf-sup condition (3.132). The main idea to verify the inf-sup condition is to find a $\eta_h^{q_h}$ for any $q_h \in Q_h$ such that $div\eta_h^{q_h} = q_h$ in each coarse element K and $\|\eta_h^{q_h}\|_{H(div,\Omega)} \leq C\|q_h\|_{0,\Omega}$. One can show that the constant C depends on global fields u_1,..., u_N and uniformly bounded via a constructive argument. Once we have the inf-sup condition, we can follow the proof of Theorem 3.3.12 to obtain the convergence rate

$$\|u - u_h\|_{H(div,\Omega)} + \|p - p_h\|_{0,\Omega} \leq C(h^{\min\{\alpha+\beta_1-\beta_2-1,1\}} + \delta).$$

3.4. Numerical Results

In this section, we first briefly introduce the discretization of saturation equations of two-phase flows. Then numerical results are presented for two-phase flow simulations.

3.4.1. Discretization of Saturation Equation

Before we present numerical results for two-phase flows, we would like to discuss the discretization of the saturation equation. We use the two-phase flow equation defined in Section 2.1 to discuss the time discretiszation, i.e.,

$$\begin{aligned} -div(\lambda(S)k\nabla p) &= f \\ \frac{\partial S}{\partial t} + div(F) &= 0 \end{aligned} \tag{3.143}$$

where $F = uf_w(S) = (-\lambda(S)k\nabla p)f_w(S)$. It is often to use IMPES (implicit pressure explicit saturation) scheme or improved IMPES scheme (c.f. Chapter 7 in [23]) to solve the coupled system (3.143). In this subsection, we briefly introduce some discretization techniques for the saturation equation, i.e., the second equation defined in Section 3.143.

Let Γ_{ij} be the common face (or edge) of K_i and K_j and n_{ij} be the normal vector pointing from K_i to K_j. Using the θ-rule for temporal discretization and a finite-volume scheme for the saturation equation, it follows the following form.

$$\frac{1}{\Delta t}(S_i^{n+1} - S_i^n) + \frac{1}{|K_i|}\sum_{j\neq i}[\theta F_{ij}(S^{n+1}) + (1-\theta)F_{ij}(S^n)] = 0, \tag{3.144}$$

where S_i^n is the cell-average of water saturation at $t = t_n$, i.e.,

$$S_i^n = \langle S(x,t_n)\rangle_{K_i}$$

and F_{ij} is a numerical approximation of the flux over Γ_{ij}, i.e.,

$$F_{ij}(S) \approx \int_{\Gamma_{ij}} f_w(S)_{ij} u_{ij} \cdot n_{ij} ds.$$

We note that no summation convention is used in the last equation. Here $f_w(S)_{ij}$ denotes the fractional-flow function associated with Γ_{ij} and the first-order upstream weighting scheme for it is defined as

$$f_w(S)_{ij} = \begin{cases} f_w(S_i) & \text{if } u \cdot n_{ij} \geq 0 \\ f_w(S_j) & \text{if } u \cdot n_{ij} < 0. \end{cases}$$

For $\theta = 0$ or 1, we can write (3.144) as a vector form

$$S^{n+1} = S^n + (\delta_x^t)^T A f(S^m), \quad m = n \quad or \quad n+1,$$

where $(\delta_x^t)_i = \frac{\Delta t}{|K_i|}$.

If $\theta = 0$, then (3.144) is an explicit scheme and only stable provided that time step Δt satisfies a stability condition (CFL) condition, i.e.,

$$\Delta t \leq \frac{|K_i|}{v_i^{in} \max_{0 \leq s \leq 1}\{f'(s)\}},$$

where v_i^{in} is the inflow flux on K_i.

For $\theta = 1$, (3.144) is an implicit scheme and unconditionally stable but gives rise to a nonlinear system. Such a nonlinear system is often solved with a Newton or Newton-Raphson iterative method. Define

$$G(S^{n+1}) = S^{n+1} - S^n - (\delta_x^t)^T A f(S^{n+1}). \tag{3.145}$$

By Taylor expansion, we have

$$G(S^{n+1}) \approx G(S^n) + G'(S^n)(S^{n+1} - S^n).$$

Noticing $G(S^{n+1}) = 0$ we have $\delta S^n := S^{n+1} - S^n = -[G'(S^n)]^{-1} G(S^n)$. From (3.145), we have

$$G'(S) = I - (\delta_x^t)^T A f'(S),$$

where $f'(S)_i = f'(S_i)$. Hence

$$S^{n+1} = S^n + \delta S^n.$$

This iteration proceeds until the norm of δS^n is smaller than a prescribed value. For other schemes of saturation equation, refer to [23].

3.4.2. Numerical Results Using Single Global Information

In this subsection, we present numerical results for permeability fields from SPE Comparative Solution Project [24] (also known as SPE 10). These permeability fields, as it was mentioned earlier, have channelized structure and a large aspect ratio. We will show that if one takes use of limited global information based on single-phase flow information in constructing multiscale basis functions, then the numerical approximation on the coarse grid becomes much more accurate.

In our numerical simulations, we will perform two-phase flow and transport simulations defined in Section 2.1. In our simulations, we take $k_{rw}(S) = S^2$ and $k_{ro}(S) = (1 - S)^2$. In the presence of capillary effects, an additional diffusion term is present in (2.3). In the simulations, we solve the pressure equation on the coarse grid and re-construct the fine-scale velocity field which is used to solve the saturation equation. The basis functions are constructed at time zero and not changed throughout the simulations.

In our numerical results, we compare the saturation fields and water-cut data as a function of pore volume injected (PVI). The water-cut is defined as the fraction of water in the produced fluid and is given by q_w/q_t, where $q_t = q_o + q_w$, with q_o and q_w being the flow rates of oil and water at the production edge of the model. In particular, $q_w = \int_{\partial\Omega^{out}} f(S)v \cdot nds$, $q_t = \int_{\partial\Omega^{out}} v \cdot nds$, where $\partial\Omega^{out}$ is the outer flow boundary. Pore volume injected, defined as $PVI = \frac{1}{V_p}\int_0^t q_t(\tau)d\tau$, with V_p being the total pore volume of the system, provides the dimensionless time for the displacement. The permeability field $k(x)$ is given on 60×220 fine grid and coarse grid is used in two-phase flow simulations without updating basis functions. We consider a traditional five-spot problem (e.g., [1]), where the water is injected at right top corner and oil is produced at four corners of the rectangular domain.

We implement the numerical simulation on 6×10, 12×11 and 12×22 coarse grid at layer 40, 50 and 70 respectively. On the three different coarse grids and the three different layers, we compare the relative water-cut error (in L^1) and the relative saturation error (in L^2) by utilizing the local mixed MsFEM and the mixed MsFEM using limited global information from single phase flow. Table 3.1, table 3.2 and table 3.3 list the relative water-cut error and relative saturation error on the different coarse grids at at layer 40, 50 and 70 respectively when $\frac{\mu_w}{\mu_o} = \frac{1}{3}$. Table 3.4, table 3.5 and table 3.6 list the relative water-cut error and relative saturation error on the different coarse grids at at layer 40, 50 and 70 respectively when $\frac{\mu_w}{\mu_o} = \frac{1}{10}$.

In Figure 3.3, the saturation profiles at $PVI = 1$ for layer 50 are compared. The simulations are run with 12×11 coarse grid and $\frac{\mu_w}{\mu_o} = 1/3$. Figure 3.4 describes the relative saturation error from $PVI = 0$ to $PVI = 1$. We observe from these figures that the saturation by using global mixed MsFEM provides an accurate representation of the reference saturation. Figure 3.5 provides the comparison of water-cut curve from the reference, local mixed MsFEM and global mixed MsFEM when PVI is from 0 to 1. We observe that there is almost no difference in water-cut curve between reference and global mixed MsFEM. These observations are consistent for all other

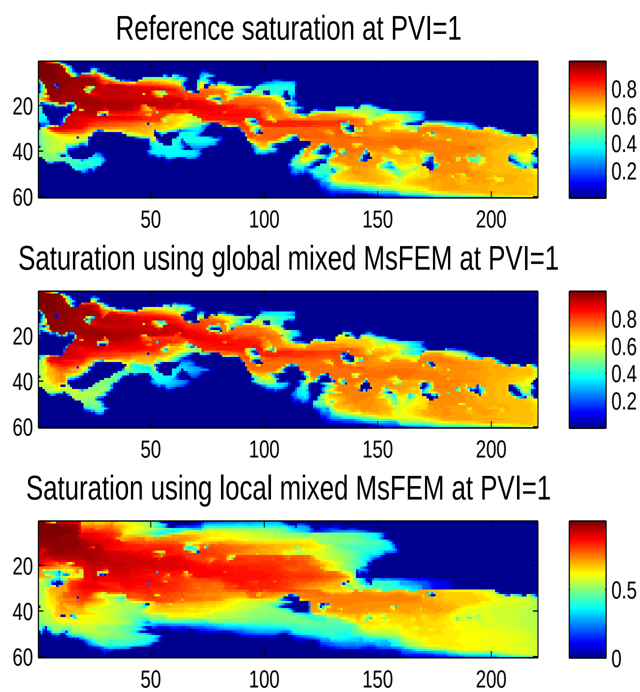

Fig. 3.3. Comparison of saturation between reference solution and MsFEM solution at $PVI = 1$, layer=50, 12×11 coarse grid and $\frac{\mu_w}{\mu_o} = 1/3$; Top: The reference saturation; Middle: The saturation using global mixed MsFEM; Bottom: Multiscale saturation using local mixed MsFEM.

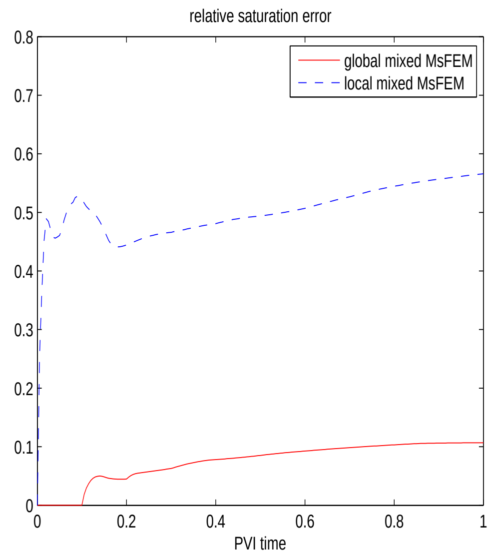

Fig. 3.4. Relative saturation error, layer=50, 12×11 coarse grid and $\frac{\mu_w}{\mu_o} = 1/3$

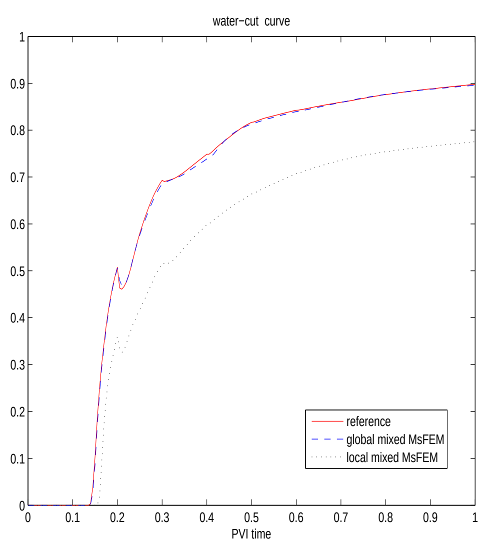

Fig. 3.5. Water-cut curve, layer=50, 12×11 coarse grid and $\frac{\mu_w}{\mu_o} = 1/3$

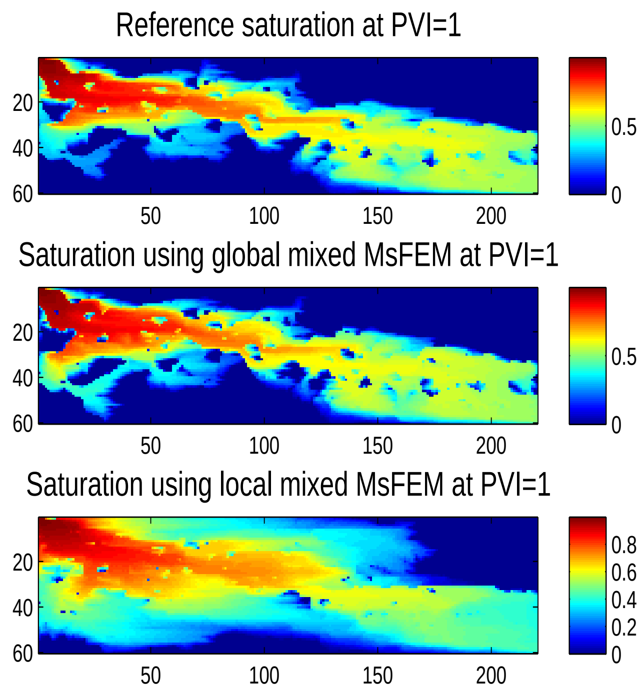

Fig. 3.6. Comparison of saturation between reference solution and MsFEM solution at $PVI = 1$, layer=50, 12×11 coarse grid and $\frac{\mu_w}{\mu_o} = 1/10$; Top: The reference saturation; Middle: The saturation using global mixed MsFEM; Bottom: Multiscale saturation using local mixed MsFEM.

layers, coarse grids and other ratios $\frac{\mu_w}{\mu_o}$. For example, Figures 3.6, 3.7 and 3.8 provide the similar saturation profile at $PVI = 1$, saturation errors and water-cut curves for $\frac{\mu_w}{\mu_o} = 1/10$.

It is clear from these figures that the use of the global information in mixed multiscale finite element methods gives us an more accurate approximation. The presented numerical results show that one can use a single phase flow solution to construct basis functions that can be employed for solving two-phase flow and transport on the coarse grid accurately. We would like to note that similar ideas have been used in ensemble level of upscaling in two-phase flow simulation [20].

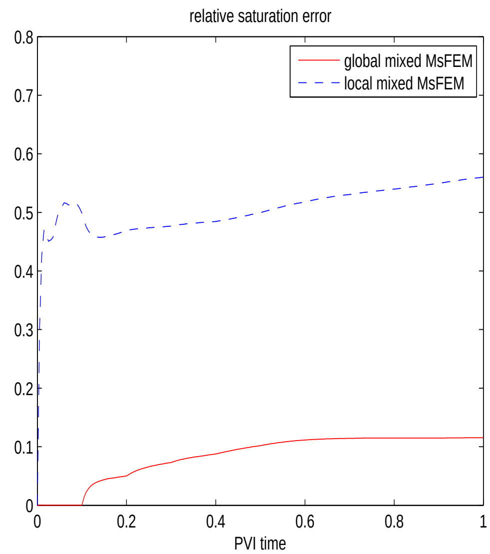

Fig. 3.7. Relative saturation error, layer=50, 12×11 coarse grid and $\frac{\mu_w}{\mu_o} = 1/10$

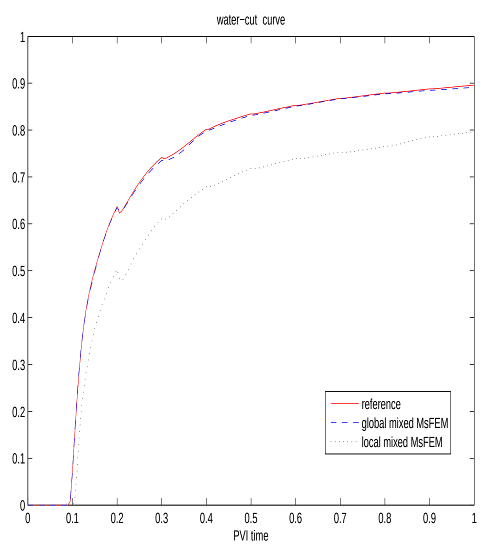

Fig. 3.8. Water-cut curve, layer=50, 12×11 coarse grid and $\frac{\mu_w}{\mu_o} = 1/10$

Next, we discuss the convergence of global mixed MsFEM with limited global information. For this reason, we consider different coarse grids, 6×10, 12×11, and 12×22 for the previous examples. As our convergence analysis indicates that the proposed method converges upto a small parameter δ which represents how well two-phase velocity field can be approximated by single-phase velocity field in each coarse patch. Table 3.1-table 3.6 show that as the coarse mesh size decreases the error decreases. This confirms our convergence analysis for the global MsFEM. We note that this is in contrast to standard MsFEM where one can observe the resonance error in the form ϵ/h. As a result, the standard mixed MsFEM does not converge as h approaches to zero.

Table 3.1. Relative Errors (layer=40, $\frac{\mu_w}{\mu_0} = 1/3$)

coarse grid	water-cut error (global Msfem)	sat. error (global Msfem)	water-cut error (local Msfem)	sat. error (local Msfem)
6×10	0.0144	0.0512	0.1172	0.2755
12×11	0.0093	0.0435	0.2057	0.3459
12×22	0.0039	0.0370	0.1867	0.3158

Table 3.2. Relative Errors (layer=50, $\frac{\mu_w}{\mu_0} = 1/3$)

coarse grid	water-cut error (global Msfem)	sat. error (global Msfem)	water-cut error (local Msfem)	sat. error (local Msfem)
6×10	0.0129	0.0871	0.1896	0.5061
12×11	0.0055	0.0753	0.1806	0.5032
12×22	0.0046	0.0568	0.1702	0.4578

Table 3.3. Relative Errors (layer=70, $\frac{\mu_w}{\mu_0} = 1/3$)

coarse grid	water-cut error (global Msfem)	sat. error (global Msfem)	water-cut error (local Msfem)	sat. error (local Msfem)
6 × 10	0.0106	0.0562	0.0408	0.2291
12 × 11	0.0081	0.0483	0.0863	0.2858
12 × 22	0.0039	0.0421	0.0976	0.2530

Table 3.4. Relative Errors (layer=40, $\frac{\mu_w}{\mu_0} = 1/10$)

coarse grid	water-cut error (global Msfem)	sat. error (global Msfem)	water-cut error (local Msfem)	sat. error (local Msfem)
6 × 10	0.0080	0.0534	0.0902	0.2721
12 × 11	0.0056	0.0491	0.1382	0.3472
12 × 22	0.0026	0.0403	0.1414	0.3153

Table 3.5. Relative Errors (layer=50, $\frac{\mu_w}{\mu_0} = 1/10$)

coarse grid	water-cut error (global Msfem)	sat. error (global Msfem)	water-cut error (local Msfem)	sat. error (local Msfem)
6 × 10	0.0049	0.0957	0.1577	0.5137
12 × 11	0.0042	0.0850	0.1499	0.5063
12 × 22	0.0041	0.0628	0.1404	0.4613

Table 3.6. Relative Errors (layer=70, $\frac{\mu_w}{\mu_0} = 1/10$)

coarse grid	water-cut error (global Msfem)	sat. error (global Msfem)	water-cut error (local Msfem)	sat. error (local Msfem)
6×10	0.0044	0.0629	0.0280	0.2262
12×11	0.0027	0.0522	0.0576	0.2736
12×22	0.0025	0.0473	0.0678	0.2397

3.4.3. Numerical Results Using Multiple Global Information

In this subsection, we present numerical results for parameter dependent permeability fields. We will show that if one takes a larger set of global fields in constructing multiscale basis functions, then the numerical approximation on the coarse grid becomes more accurate. For our numerical results, we consider the following permeability field,

$$k(x, \theta) = \exp(\theta_i Y_i(x)),$$

where $Y_i(x)$ is heterogeneous permeability fields and $\theta_i \in R$. As for $Y_i(x)$, we use heterogeneous permeability fields from SPE Comparative Solution Project [24] (also known as SPE 10). These permeability fields have channelized structure and a large aspect ratio. Because of channelized structure of the permeability fields, the localized approaches do not perform well. On the other hand, it is known [29] that the use of limited global information based on single-phase flow information improves the accuracy. Because of $\theta = (\theta_1, ..., \theta_M)$ dependence, one needs to use several single-phase flow solutions. Our goal here is to choose a few permeability realizations such that using these realizations, one can construct basis functions which can be used to approximate the solution for any realization (any θ). Because of smooth dependence of permeability with respect to θ, it can be shown [27] that mixed multiscale finite element method converges independent of small scales if sufficient number of realizations are chosen.

In our numerical experiments, we consider $k(x,\theta) = \exp(\theta Y(x))$. We consider a range of θ, $\theta_1 \le \theta \le \theta_2$, and use the global single-phase flow solutions corresponding to end points $\theta = \theta_1$ and $\theta = \theta_2$ to construct the multiscale basis functions. In particular, $u_1 = -k(x,\theta_1)\nabla p(x,\theta_1)$ and $u_2 = -k(x,\theta_2)\nabla p(x,\theta_2)$ are used to construct mixed multiscale basis functions as described earlier.

In our numerical simulations, we will perform two-phase flow and transport simulations defined in Section 2.1. In our simulations, we take $k_{rw}(S) = S^2$ and $k_{ro}(S) = (1-S)^2$. In the presence of capillary effects, an additional diffusion term is present in (2.3). In the simulations, we solve the pressure equation on the coarse grid and re-construct the fine-scale velocity field which is used to solve the saturation equation. The basis functions are constructed at time zero and not changed throughout the simulations.

In our numerical results, we compare the saturation fields and water-cut data as a function of pore volume injected (PVI). The permeability field $Y(x)$ is given on 60×220 fine grid and 6×22 coarse grid is used in two-phase flow simulations without updating basis functions. We consider a traditional five-spot problem (e.g., [1]), where the water is injected in the middle and oil is produced at four corners of the rectangular domain.

In Figure 3.9, the water-cut and the saturation profiles for a value of $\theta = 0.75$ are compared. The global fields corresponding to single-phase flow solutions are computed at $\theta_1 = 0.5$ and $\theta_2 = 1$. The simulations are run with $\mu_o/\mu_w = 5$. We note that the value of θ is different from the values used in generating basis functions. We observe from these figures that the mixed multiscale finite element method provides an accurate representation of the solution. In particular, there is almost no difference in water-cut curve and the error in the saturation profile at $PVI = 1$ is less than 5 %. This observation is consistent for all other values between θ_1 and θ_2, and it is demonstrated next.

In our next set of numerical experiments, water-cut errors and saturation errors for values of θ between $\theta_1 = 0.5$ and $\theta_2 = 1.5$ are presented. We also compare these results with the results obtained using only one value of θ, $\theta = 1$. More precisely, we only use the global solution corresponding to $\theta = 1$ to construct multiscale basis functions. Furthermore, these basis functions are used for solving two-phase flow and transport on the coarse grid for other values of θ. We observe from Figures 3.10 - 3.11, that the results are substantially better if two global solutions are employed in characterizing the solutions for all the range of θ. In Figure 3.10, $\mu_o/\mu_w = 0.1$ is taken and in Figure 3.11, $\mu_o/\mu_w = 10$ is taken. It is clear from these figures that the use of two global solutions in mixed multiscale finite element methods gives us an accurate approximation. The presented numerical results show that one can use a few realizations of the permeability field to construct basis functions that can be employed for solving two-phase flow and transport on the coarse grid accurately. We would like to note that similar ideas have been used in ensemble level of upscaling in two-phase flow simulation [20]. The proposed approach has now been applied to more complicated stochastic permeability fields in [2] where the uncertainty is parameterized via high dimensional parameter θ (i.e., $M > 1$). Using a few values of θ, one can pre-compute multiscale basis functions that are used to characterize the uncertainty.

One of our goals with presented numerical results is to show that the solution can be approximated using multiple global fields. Next, we discuss the convergence of global mixed MsFEM with limited global information. For this reason, we consider different coarse grids, 6×22, 12×44, and 15×55 for the previous example with

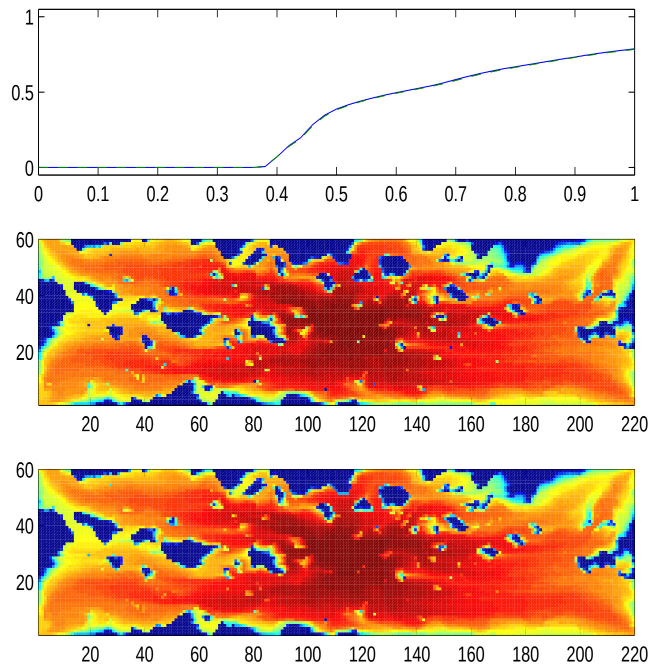

Fig. 3.9. Top: Comparison of water-cut between reference solution and multiscale solution; Middle: The reference saturation at $PVI = 1$; Bottom: Multiscale saturation at $PVI = 1$. (layer 85)

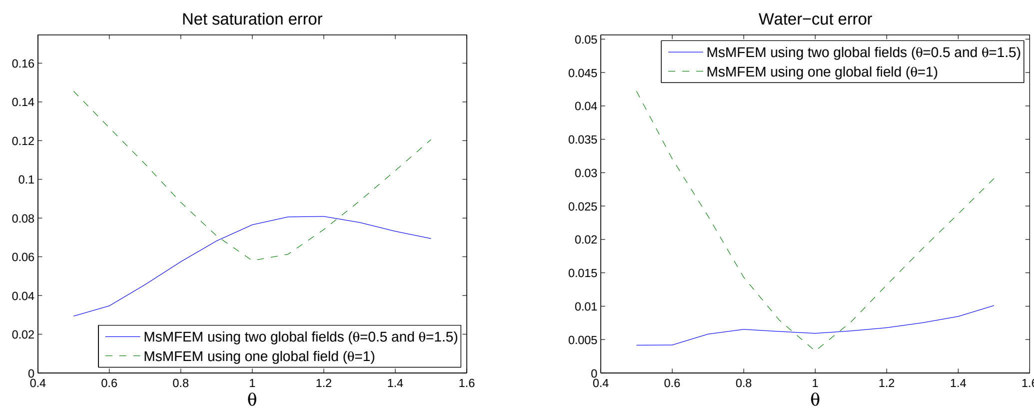

Fig. 3.10. The saturation and water-cut error using one single-phase flow solution and two single-phase flow solutions, $\mu_o/\mu_w = 0.1$. (layer 85)

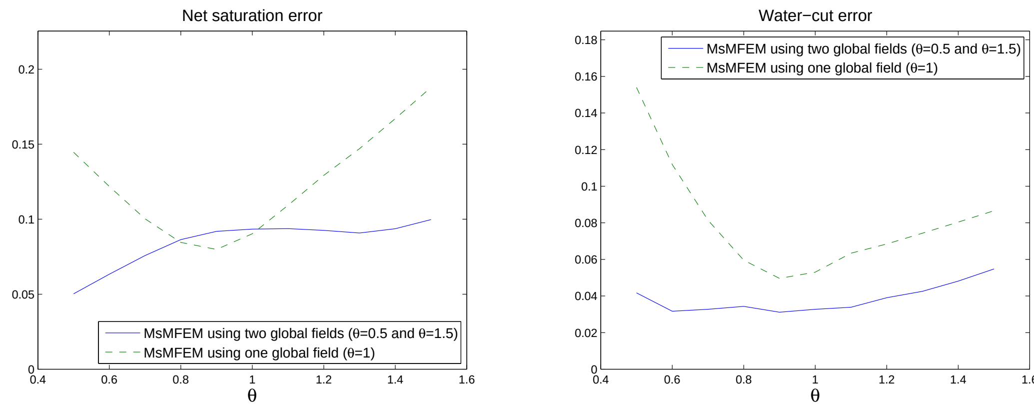

Fig. 3.11. The saturation and water-cut error using one single-phase flow solution and two single-phase flow solutions, $\mu_o/\mu_w = 10$. (layer 85)

$\mu_o/\mu_w = 10$. As our convergence analysis indicates that the proposed method converges upto a small parameter δ which represents how well two-phase velocity field can be approximated by single-phase velocity field in each coarse patch. Moreover, the convergence rate also depends on the smoothness of A_i in (3.117). One can consider an ideal toy problem where the convergence rate can be verified by specifying the form of the solution upto smooth functions A_i (see (3.117). Instead, we would like to consider SPE 10 example and show that as the coarse mesh size decreases the error decreases. We note that this is in contrast to standard MsFEM where one can observe the resonance error in the form ϵ/h. As a result, the mixed MsFEM does not converge as h approaches to zero. As we see from Figure 3.12, the mixed MsFEM using limited global information converges as the coarse mesh size decreases. This is an indication that for general complicated media such as SPE 10 with high contrast, one can expect the convergence of mixed MsFEM using limited global information. As we mentioned earlier that if one uses directional fields, then $\delta = 0$. In this case, one can perhaps show more rigorous convergence rates in the absence of source terms.

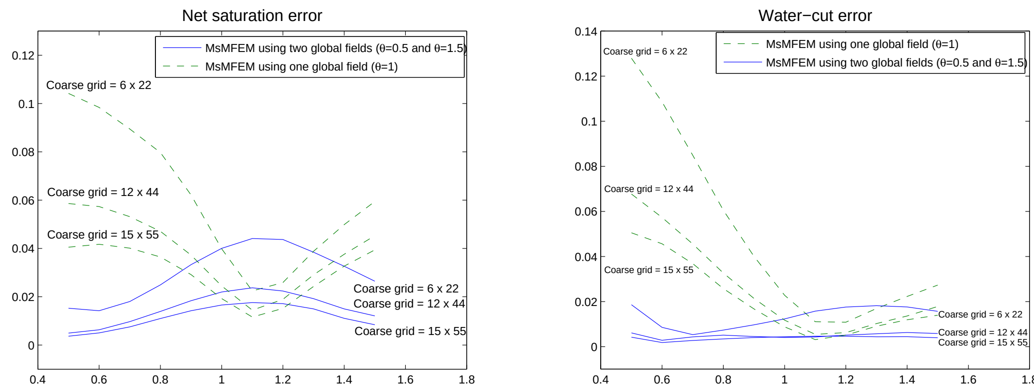

Fig. 3.12. The saturation and water-cut error using one single-phase flow solution and two single-phase flow solutions, $\mu_o/\mu_w = 10$ for different degree of coarsening. (layer 85)

3.5. Conclusions and Comments

In this chapter, we present the multiscale numerical methods: Galerkin MsFEM, PUM, and mixed MsFEM. These methods employ limited global information. In multi-phase flow simulations, the global fields are employed from the solutions of single-phase flow problems to construct multiscale basis functions. Our analysis assumes the solution of partial differential equation smoothly depends on the global information. Under this assumption, we derive convergence rate for the global multiscale numerical methods. This assumption is relaxed for the case with scale separation, where we show that multiscale finite element methods converge without the resonance error. The convergence analysis shows that approximation is improved and accurate compared to local MsFEMs.

A few numerical results are shown. First we consider two-phase flows simulation. We consider complicated spatial heterogeneities presented in SPE Comparative Project [24]. Single-phase flow solution is used to construct velocity basis functions. Secondly, we consider a parameter dependent permeability field (a simplified case for general stochastic permeability fields) where a limited number of parameter values are used to generate the global fields. Based on these global fields, we compute multiscale basis functions. These permeability fields (SPE 10) are channelized and difficult to upscale and, thus, single-phase flow information (limited global information) is used for two-phase flow simulations. Our numerical results show that one can accurately approximate the solution of parameter dependent two-phase flow equations with a few global fields corresponding to single-phase flow solutions. One can find a few more representative numerical examples in [3] involving highly channelized permeability as well as a 3-D reservoir model with unstructured fine grid. We observe that the use of unstructured coarse grids provides more accurate results and more flexibility.

CHAPTER IV

MULTISCALE NUMERICAL METHODS FOR PARABOLIC EQUATIONS WITH CONTINUUM SCALES USING GLOBAL INFORMATION

In this chapter, we consider multiscale approaches for solving parabolic equations with heterogeneous coefficients. Our interest stems from porous media applications and we assume that there is no scale separation with respect to spatial variables. To compute the solution of these multiscale problems on a coarse grid as before, we define global fields such that the solution smoothly depends on these fields. We present various finite element discretization techniques and provide analyses of these methods. A few representative numerical examples are presented using heterogeneous fields with strong non-local features. These numerical results demonstrate that the solution can be captured accurately on the coarse grid when some type of limited global information is used for parabolic equations. The results of this chapter can been found in our papers [5, 45].

The chapter is organized as follows.

In Section 4.1, we present problem setting. In Section 4.2, we present an analysis of MsFEM with single global information for parabolic equations. In Section 4.3, we present the analysis for general case using PUM with multiple global fields. In Section 4.4, we discuss the global mixed MsFEM for parabolic equations. In Section 4.5, we present a few numerical results for Richards equations. Finally, we draw some conclusions and make further comments.

4.1. Preliminaries on Parabolic Equations

In this chapter, we consider a model parabolic equation

$$\begin{aligned} D_t p - div k \nabla p &= f \quad \text{in} \quad \Omega_T \\ p(t=0) &= p_0 \quad \text{in } \Omega \end{aligned} \tag{4.1}$$

where $\Omega_T := \Omega \times [0, T)$ and $k := k(x,t)$ is uniformly positive and bounded in Ω_T, $k(x,t)$ is a rough function with respect to x, $f = f(x,t) \in L^2(L^2(\Omega))$ and the problem is subject to boundary conditions. Our objective is to define multiscale methods that can capture the solution of (4.1) when $k(x,t)$ does not have scale separation with respect to spatial variable. This type of problems arise in many porous media applications, e.g., in compressible flow, Richards equations and etc.

Next, we introduce some notations. Let M be a finite dimensional subspace of H_0^1 and for simplicity, we consider $p|_{\partial\Omega} = 0$. We suppose that $p_h : [0, T) \longmapsto M$ solve the following equation

$$\begin{aligned} (D_t p_h, v_h) + (k \nabla p_h, \nabla v_h) &= (f, v_h) \\ (p_h(0), v_h^l) &= (p_0, v_h^l) \end{aligned} \tag{4.2}$$

for all $v_h \in M$ and $v_h^l \in S^h$ (or M and it depends on the feature of initial values), where S_h denotes standard polynomial finite element spaces. Let $p_{0,h} = (p_0, v_h^l)$. Let us define in this chapter

$$|||p - p_h|||_{\Omega_T}^2 = \|p - p_h\|_{L^\infty(L^2(\Omega))}^2 + |p - p_h|_{L^2(H^1(\Omega))}^2.$$

The following is a known stability result [36].

Lemma 4.1.1. *Let p and p_h be the solutions to (4.1) and (4.2) respectively. Assume* *that* $w : [0, T) \longmapsto M$ *be any function, then*

$$|||p - p_h|||_{\Omega_T} \leq C(|||p - w|||_{\Omega_T} + \|D_t(p - w)\|_{L^2(H^{-1}(\Omega))} + \|p_{0,h} - w(0)\|_{L^2(\Omega)}). \quad (4.3)$$

Proof. Let $\xi = p_h - w$ and $\eta = p - w$, then for $v \in M$

$$(D_t w, v) + (k\nabla w, \nabla v) = (f, v) + (k\nabla w - k\nabla p, \nabla v) - (D_t \eta, v).$$

Take $v = \xi$, then

$$(D_t \xi, \xi) + (k\nabla \xi, \nabla \xi) = (k\nabla u - k\nabla w, \nabla \xi) + (D_t \eta, \xi).$$

Consequently,

$$\begin{aligned} D_t \|\xi\|_{0,\Omega}^2 + |\xi|_{1,\Omega}^2 &\leq \frac{C_1}{2\gamma_1}|\eta|_{1,\Omega}^2 + \frac{\gamma_1}{2}|\xi|_{1,\Omega}^2 + \frac{1}{2\gamma_2}\|\xi\|_{1,\Omega}^2 + \frac{\gamma_2}{2}\|D_t\eta\|_{-1,\Omega}^2 \\ &\leq \frac{C_1}{2\gamma_1}|\eta|_{1,\Omega}^2 + (\frac{\gamma_1}{2} + \frac{1}{2\gamma_2})|\xi|_{1,\Omega}^2 + \frac{1}{2\gamma_2}\|\xi\|_{0,\Omega}^2 + \frac{\gamma_2}{2}\|D_t\eta\|_{-1,\Omega}^2 \end{aligned}$$

Pick proper values of γ_1 and γ_2 so that

$$\|\xi\|_{0,\Omega}^2(\tau) + \int_0^\tau |\xi|_{1,\Omega}^2(s)ds \leq C(|\eta|_{L^2(H^1(\Omega))}^2 + \|D_t\eta\|_{L^2(H^{-1}(\Omega))}^2 + \|\xi(0)\|_{0,\Omega}^2) + \int_0^\tau \|\xi\|_{0,\Omega}^2(s)ds,$$

and Gronwall's inequality (i.e., Lemma 2.2.7) implies that

$$\|\xi\|_{L^\infty(L^2(\Omega))}^2 + |\xi|_{L^2(H^1(\Omega))} \leq C(|\eta|_{L^2(H^1(\Omega))}^2 + \|D_t\eta\|_{L^2(H^{-1}(\Omega))}^2 + \|\xi(0)\|_{0,\Omega}^2).$$

Since $p - p_h = \eta - \xi$, the triangle inequality gives

$$\|p - p_h\|_{L^\infty(L^2(\Omega))} + |p - p_h|_{L^2(H^1(\Omega))}$$

$$\leq C(\|p - w\|_{L^\infty(L^2(\Omega))} + |p - w|_{L^2(H^1(\Omega))} + \|D_t(p - w)\|_{L^2(H^{-1}(\Omega))} + \|p_{0,h} - w(0)\|_{0,\Omega}).$$

$\square$

Lemma 4.1.1 implies the following inequality

$$|||p - p_h|||_{\Omega_T} \leq C(|||p - w|||_{\Omega_T} + \|D_t(p - w)\|_{L^2(L^2(\Omega))} + \|p_{0,h} - w(0)\|_{L^2(\Omega)}). \quad (4.4)$$

We note that *summation convention* is used in this chapter.

4.2. Galerkin MsFEM with Limited Global Information

The key idea of the method has been described in Section 3.2.1. The basis functions, $\phi_i^K, i = 1, \cdots, d$ are the set to satisfy the elliptic problem (3.4) Let p_1 be the solution of the following equation

$$-divk(x)\nabla p_1 = f_0 \quad \text{in} \quad \Omega, \qquad (4.5)$$

subject to the same boundary conditions as the original equation. Here f_0 is time independent right hand side and can be thought as the heterogeneous spatial part of f, e.g., $f(x, t) = f_0(x)g(t)$. If $f \in L^2(L^2(\Omega))$, we can take $f_0 = 0$ or any smooth function because the small scale features of the solution, p, are independent of the source term provided the source term is a smooth function. In simulations, (4.5) is solved on the fine-scale grid by using standard finite element method, e.g., using piecewise linear basis functions. We note that this is one time overhead, though these basis functions capture the non-local information and allows us to obtain the convergence independent of small scales.

We define the Galerkin finite element space by

$$V_h = span\{\phi_i^K : 1, \cdots, d; K \in \tau_h\}. \qquad (4.6)$$

The weak formulation of (4.1) is to seek $p_h \in V_h \subset H_0^1$ such that

$$\begin{aligned}(D_t p_h, v_h) + (k\nabla p_h, \nabla v_h) &= (f, v_h)\\ (p_h(0), v_h^l) &= (p_0, v_h^l)\end{aligned} \tag{4.7}$$

for all $v_h \in V_h$ and $v_h^l \in S^h$ (or V_h).

For the analysis, we may assume that p smoothly depends on p_1. This can be derived for a channelized media as it is done in [29].

Assumption G. There exists a sufficiently smooth scalar valued function $G(\eta, t)$ $(G \in L^\infty(H^2) \cap H^1(H^2) \cap L^{\frac{2s}{s-2}}(W^{3,\frac{2s}{s-2}}), s > 2)$*, such that*

$$|||p - G(p_1, t)|||_{\Omega_T} + ||D_t(p - G(p_1, t))||_{L^2(L^2(\Omega))} \leq C\delta, \tag{4.8}$$

where δ *is sufficiently small.*

Note that G is defined on a bounded domain in R^2 because p is a bounded function and $[0, T]$ is bounded.

To get an accurate numerical solution of equation (4.2) on the coarse grid, we use the global solutions (on the fine grid) of equation (4.5) to construct MsFEM basis functions as described earlier. Next, we prove that with these basis functions and *Assumption G*, MsFEM converges independent of small scales.

Theorem 4.2.1. *Under Assumption G and* $p \in W^{1,s}(\Omega)$ $(s > 2)$*, we have*

$$|||p - p_h|||_{\Omega_T} \leq C\delta + C\alpha(h) + Ch^{1-\frac{2}{s}}|p_1|_{1,s,\Omega},$$

where $\alpha(h)$ *is the approximation error for initial values.*

Proof. In the proof, we first use Lemma 4.1.1 and then break the estimate in this lemma into the estimates over each coarse grid block. The estimation over each coarse grid block is obtained using perturbation of G and regularities of basis functions and the global field. Following standard practice of finite element estimation, we seek finite dimensional function $w = c_i(t)\phi_i^K$, where ϕ_i^K are specified in (**??**). Our analysis is similar to that presented in [4] for elliptic equations.

From Lemma 4.1.1, we have

$$
\begin{aligned}
|||p - p_h|||_{\Omega_T} \leq C(&|||p - G(p_1,t)|||_{\Omega_T} + |||G(p_1,t) - c_i(t)\phi_i^K|||_{\Omega_T} \\
&+ \|D_t(p - G(p_1,t))\|_{L^2(L^2(\Omega))} + \|D_t(G(p_1,t) - c_i(t)\phi_i^K)\|_{L^2(L^2(\Omega))} \\
&+ \|p_{0,h} - c_i(0)\phi_i^K\|_{L^2(\Omega)}).
\end{aligned} \tag{4.9}
$$

We may assume that $\|p_{0,h} - c_i(0)\phi_i^K\|_{L^2(\Omega)}) \leq \alpha(h)$. Hence,

$$
\begin{aligned}
|||p - p_h|||_{\Omega_T} \leq C\delta + C\alpha(h) + C(&|||G(p_1,t) - c_i(t)\phi_i^K|||_{\Omega_T} \\
&+ \|D_t(G(p_1,t) - c_i(t)\phi_i^K)\|_{L^2(L^2(\Omega))}).
\end{aligned} \tag{4.10}
$$

Next, we present an estimate for the third term. We choose $c_i(t) = G(p_1(x_i),t)$ for $t > 0$, where x_i are vertices of K. Furthermore, using Taylor expansion of G around $\overline{p_1}_K$, which is the average of p_1 over K,

$$
\begin{aligned}
G(p_1(x_i),t) = &G(\overline{p_1}_K,t) + \partial_\eta G(\overline{p_1}_K,t)(p_1(x_i) - \overline{p_1}_K) \\
&+ (p_1(x_i) - \overline{p_1}_K)^2 \int_0^1 s\partial_\eta^2 G(p_1(x_i) + s(\overline{p_1}_K - p_1(x_i)),t)ds,
\end{aligned}
$$

where $\partial_\eta^2 G$ refers to the second derivative with respect to first variable $G(\eta,t)$, We have in each K

$$
\begin{aligned}
c_i(t)\phi_i^K = &G(\overline{p_1}_K,t)\sum_i \phi_i^K + \partial_\eta G(\overline{p_1}_K,t)(p_1(x_i) - \overline{p_1}_K)\phi_i^K \\
&+ (p_1(x_i) - \overline{p_1}_K)^2\phi_i^K \int_0^1 s\partial_\eta^2 G(p_1(x_i) + s(\overline{p_1}_K - p_1(x_i)),t)ds \\
&= G(\overline{p_1}_K,t) + \partial_\eta G(\overline{p_1}_K,t)(p_1(x_i)\phi_i^K - \overline{p_1}_K) \\
&+ (p_1(x_i) - \overline{p_1}_K)^2\phi_i^K \int_0^1 s\partial_\eta^2 G(p_1(x_i) + s(\overline{p_1}_K - p_1(x_i)),t)ds.
\end{aligned} \tag{4.11}
$$

In the last step, we have used $\sum_i \phi_i^K = 1$. Similarly, in each K,

$$\begin{aligned} G(p_1(x),t) &= G(\overline{p_1}_K,t) + \partial_\eta G(\overline{p_1}_K)(p_1(x) - \overline{p_1}_K) \\ &\quad + (p_1(x) - \overline{p_1}_K)^2 \int_0^1 s\partial_\eta^2 G(p_1(x) + s(\overline{p_1}_K - p_1(x)),t)ds. \end{aligned} \tag{4.12}$$

Using (4.11) and (4.12), we get

$$\begin{aligned} &\|G(p_1,t) - c_i(t)\phi_i^K\|_{L^\infty(L^2(K))} \\ &\le \|\partial_\eta G(\overline{p_1}_K,t)(p_1(x) - p_1(x_i)\phi_i^K)\|_{L^\infty(L^2(K))} + \\ &\|(p_1(x_i) - \overline{p_1}_K)^2\phi_i^K \int_0^1 s\partial_\eta^2 G(p_1(x_i) + s(\overline{p_1}_K - p_1(x_i)),t)ds\|_{L^\infty(L^2(K))} \\ &+ \|(p_1(x) - \overline{p_1}_K)^2 \int_0^1 s\partial_\eta^2 G(p_1(x) + s(\overline{p_1}_K - p_1(x)),t)ds\|_{L^\infty(L^2(K))} \\ &\le Ch^2\|f_0\|_{0,K} + Ch^{2-\frac{4}{s}}|p_1|_{1,s,K}^2 \\ &\le Ch^2\|f_0\|_{0,K} + Ch^{2-\frac{4}{s}}|p_1|_{1,s,\Omega}|p_1|_{1,s,K}^2 \\ &\le Ch^2\|f_0\|_{0,K} + Ch^{2-\frac{4}{s}}|p_1|_{1,s,K}. \end{aligned} \tag{4.13}$$

From the third step to the forth step, we have used the fact $|p_1|_{1,s,K}^2 \le |p_1|_{1,s,\Omega}|p_1|_{1,s,K} \le C|p_1|_{1,s,K}$ (we assume elements K's have continuous extension property). Since $p_1 \in W^{1,s}(\Omega)$ is bounded *apriori*, $|p_1|_{1,s,K}$ is absorbed into the constant C. In the second step we used the fact $\|p_1(x) - p_1(x_i)\phi_i^K\|_{0,K} \le Ch\|f_0\|_{0,K}$ (see [4]) , smoothness of $G(\eta,t)$ and the inequality

$$|p_1(x) - p_1(y)| \le C|x-y|^{1-\frac{2}{s}}|p_1|_{1,s,K},$$

for $x, y \in K$ when $s > 2$. A direct calculation gives

$$
\begin{aligned}
&|(p_1(x) - \overline{p_1}_K)^2 \int_0^1 s\partial_\eta^2 G(p_1(x) + s(\overline{p_1}_K - p_1(x)), t)ds|_{L^2(H^1(K))} \\
&\leq \|(p_1(x) - \overline{p_1}_K)^2 \nabla p_1(x) \int_0^1 (1-s)s\partial_\eta^3 G(p_1(x) + s(\overline{p_1}_K - p_1(x)), t)ds\|_{L^2(L^2(K))} \\
&+ 2\|(p_1(x) - \overline{p_1}_K)\nabla p_1(x) \int_0^1 s\partial_\eta^2 G(p_1(x) + s(\overline{p_1}_K - p_1(x)), t)ds\|_{L^2(L^2(K))} \\
&\leq Ch^{2-\frac{4}{s}} |p_1|_{1,s,K}^2 (T|p_1|_{1,s,\Omega}^s + |G|_{L^{\frac{2s}{s-2}}(W^{3,\frac{2s}{s-2}})}^{\frac{2s}{s-2}})^{\frac{1}{2}} \\
&+ Ch^{1-\frac{2}{s}} |p_1|_{1,s,K} (T|p_1|_{1,s,\Omega}^s + |G|_{L^{\frac{2s}{s-2}}(W^{2,\frac{2s}{s-2}})}^{\frac{2s}{s-2}})^{\frac{1}{2}} \\
&\leq Ch^{2-\frac{4}{s}} |p_1|_{1,s,K}^2 + Ch^{1-\frac{2}{s}} |p_1|_{1,s,K} \\
&\leq Ch^{1-\frac{s}{2}} |p_1|_{1,s,K},
\end{aligned}
\tag{4.14}
$$

where we used Young's inequality in the second step. Noticing $|p_1(x) - p_1(x_i)\phi_i^K|_{1,K} \leq Ch\|f_0\|_{0,K}$, we get

$$
\begin{aligned}
&|G(p_1, t) - c_i(t)\phi_i^K|_{L^2(H^1(K))} \\
&\leq \|\partial_\eta G(\overline{p_1}_K, t)(p_1(x) - p_1(x_i)\phi_i^K)\|_{L^2(H^1(K))} + \\
&\|(p_1(x_i) - \overline{p_1}_K)^2 \phi_i^K \int_0^1 s\partial_\eta^2 G(p_1(x_i) + s(\overline{p_1}_K - p_1(x_i)), t)ds\|_{L^2(H^1(K))} \\
&+ \|(p_1(x) - \overline{p_1}_K)^2 \int_0^1 s\partial_\eta^2 G(p_1(x) + s(\overline{p_1}_K - p_1(x)), t)ds\|_{L^2(H^1(K))} \\
&\leq Ch\|f_0\|_{0,K} + Ch^{2-\frac{4}{s}} |p_1|_{1,s,K}^2 |\phi_i^K|_{1,K} + Ch^{1-\frac{s}{2}} |p_1|_{1,s,K} \\
&\leq Ch\|f_0\|_{0,K} + Ch^{2-\frac{4}{s}} |p_1|_{1,s,K} + Ch^{1-\frac{s}{2}} |p_1|_{1,s,K}.
\end{aligned}
\tag{4.15}
$$

In the last step, we have used the fact that $|\nabla\phi_i^K| \le C/h$. Similarly one can estimate

$$
\begin{aligned}
&\|D_t(G(p_1,t)-c_i(t)\phi_i^K)\|_{L^2(L^2(K))}\\
&\le \|\partial_t\partial_\eta G(\overline{p_1}_K,t)(p_1(x)-p_1(x_i)\phi_i^K)\|_{L^2(L^2(K))}+\\
&\|(p_1(x_i)-\overline{p_1}_K)^2\phi_i^K\int_0^1 s\partial_t\partial_\eta^2 G(p_1(x_i)+s(\overline{p_1}_K-p_1(x_i)),t)ds\|_{L^2(L^2(K))}\\
&+\|(p_1(x)-\overline{p_1}_K)^2\int_0^1 s\partial_t\partial_\eta^2 G(p_1(x)+s(\overline{p_1}_K-p_1(x)),t)ds\|_{L^2(L^2(K))}\\
&\le Ch^2\|f_0\|_{0,K}+Ch^{2-\frac{4}{s}}|p_1|_{1,s,K}^2\\
&\le Ch^2\|f_0\|_{0,K}+Ch^{2-\frac{4}{s}}|p_1|_{1,s,K}.
\end{aligned}
\tag{4.16}
$$

In the second step, we used the regularity $G \in H^1(H^2)$. Combing (4.13), (4.15) and (4.16), we have

$$
\begin{aligned}
&|||G(p_1,t)-c_i\phi_i^K|||_K+||D_t(G(p_1,t)-c_i\phi_i^K)\|_{L^2(L^2(K))}\\
&\le Ch\|f_0\|_{0,K}+Ch^{2-\frac{4}{s}}|p_1|_{1,s,K}+Ch^{1-\frac{s}{2}}|p_1|_{1,s,K}.
\end{aligned}
\tag{4.17}
$$

Summing (4.17) over K and take into account (4.10), we have

$$|||p-p_h|||_{\Omega_T} \le C\delta+C\alpha(h)+Ch^{1-\frac{2}{s}}|p_1|_{1,s,\Omega}.$$

The proof is done. □

Remark 4.2.1. *If $\nabla p_1 \in L^\infty(\Omega)$, the regularity of G can be relaxed to $G \in H^1(H^2)\cap L^2(H^3)$ and the convergence rate would be $C\delta+C\alpha(h)+Ch$.*

Remark 4.2.2. *We can relax the assumption on G if the proof is carried out using Taylor polynomials in Sobolev spaces. In particular, it is sufficient to assume $G(\eta,t) \in H^1(H^2)\cap L^\infty(H^2)$. We will study this in the next section within the framework of partition of unity method (PUM).*

Remark 4.2.3. *If equation (4.1) has small ϵ-scale feature (see[38]), the proof of the*

Theorem of 4.2.1 implies that the resonance error $O(\frac{\epsilon}{h})$ is removed by the method described in this section.

4.3. PUM for Parabolic Equations Using Multiple Global Information

In this section, PUM is applied to parabolic equations by using multiple global fields.

4.3.1. Framework of PUM for Parabolic Equations

In Section 4.2, we have studied parabolic equations when the solution smoothly depends on a single global field. In many applications, the solution may smoothly depend on multiple fields. In this section, we are going to discuss the latter, which is more general. We utilize the idea of partition of unity method (PUM) in [49] and [58] to capture the small scale information of the solution.

We define patch w_j $(j = 1, 2, ..., m)$ to be the union of elements (coarse) sharing the common vertex x_j. Let $diam(w_j) = h_j$ and $h = \max_j\{h_j\}$. We suppose that the solution p of (4.1) can be approximated by ξ_j^p on each patch w_j $(j = 1, 2, ..., m)$ and satisfies the following conditions on each patch (no summation over j),

$$\begin{aligned} \|p(t) - \xi_j^p(t)\|_{0,\omega_j}^2 &\leq C\epsilon_1(t, j)^2 \\ |p(t) - \xi_j^p(t)|_{1,\omega_j}^2 &\leq C\epsilon_2(t, j)^2 \\ \|D_t(p(t) - \xi_j^p(t))\|_{0,\omega_j}^2 &\leq C\epsilon_3(t, j)^2. \end{aligned} \tag{4.18}$$

Let ϕ_j^0 be partition of unity functions (e.g., linear hat functions) associated with the patches w_j. If we paste $\xi_j^p(t)$ with ϕ_j^0 to get w defined in Lemma 4.1.1 (i.e., $w = \phi_j^0\xi_j^p$) and follow the techniques in [12], we can obtain the following theorem which provides us with a basic estimate for the convergence of multiscale finite element method.

Theorem 4.3.1. *Let u and u_h be the solutions to (4.1) and (4.2) respectively. If (4.18) holds, then*

$$|||p - p_h|||^2_{\Omega_T} \leq \alpha(h) + C \int_0^T (\epsilon_1(t,j)^2 + \frac{\epsilon_1^2(t,j)}{h_j^2} + \epsilon_2(t,j)^2 + \epsilon_3(t,j)^2)dt, \qquad (4.19)$$

where $\alpha(h)$ is the approximation error for initial values.

Proof. For the approximation error for initial values, we assume that

$$\|(p-w)(0)\|^2_{0,\Omega} + \|p_{0,h} - w(0)\|^2_{0,\Omega} \leq \alpha(h).$$

Let ϕ_j^0 be partition of unity functions associated with the patches w_j. If we take $w = \phi_j^0 \xi_j^p$, then (refer to [12])

$$|p - w|_{1,\Omega} \leq C(\frac{\epsilon_1(t,j)^2}{h_j^2} + \epsilon_2^2(t,j))^{\frac{1}{2}},$$

and

$$\|D_t(p-w)\|_{0,\Omega} \leq C(\sum_{j=1}^m \epsilon_3^2(t,j))^{\frac{1}{2}}.$$

Set $\eta(t) = p(t) - w(t)$. Since for any $t \in [0,T]$,

$$\begin{aligned}
\|\eta(t)\|^2_{0,\Omega} &= \|\eta(0)\|^2_{0,\Omega} + \int_0^t D_s \|\eta(s)\|^2_{0,\Omega} ds \\
&= \|\eta(0)\|^2_{0,\Omega} + 2\int_0^t (\eta, D_s\eta(s))ds \\
&\leq \|\eta(0)\|^2_{0,\Omega} + \|\eta\|^2_{L^2(L^2(\Omega))} + \|D_t\eta\|^2_{L^2(L^2(\Omega))} \\
&\leq \|\eta(0)\|^2_{0,\Omega} + C\int_0^T (\epsilon_1^2(t,j) + \epsilon_3^2(t,j))dt.
\end{aligned}$$

Hence we obtain the following estimate for the partition of unity method

$$
\begin{aligned}
&\|p-p_h\|^2_{L^\infty(L^2(\Omega))}+|p-p_h|^2_{L^2(H^1(\Omega))}\\
&\le C(\|(p-w)(0)\|^2_{0,\Omega}+\|p_{0,h}-w(0)\|^2_{0,\Omega})\\
+C&\int_0^T(\epsilon_1(t,j)^2+\frac{\epsilon_1^2(t,j)}{h_j^2}+\epsilon_2(t,j)^2+\epsilon_3(t,j)^2)dt
\end{aligned}
\tag{4.20}
$$

and which is the same as (4.19). □

Remark 4.3.1. *If the patches w_j have the uniform Poincaré property (as defined in [12], page 92), one can simplify (4.19) to*

$$
|||p-p_h|||^2_{\Omega_T}\le C\alpha(h)+C\int_0^T(h_j^2\epsilon_2(t,j)^2+\epsilon_2(t,j)^2+\epsilon_3(t,j)^2)dt. \tag{4.21}
$$

Remark 4.3.2. *By the proof of Theorem 4.3.1, (4.19) can be reduced to*

$$
|||p-p_h|||^2_{\Omega_T}\le C\alpha(h)+\sup_t\sum_{j=1}^m\epsilon_1(t,j)^2+C\int_0^T(\frac{\epsilon_1^2(t,j)}{h_j^2}+\epsilon_2(t,j)^2+\epsilon_3(t,j)^2)dt. \tag{4.22}
$$

Remark 4.3.3. *Assume the finite space $M\in P_k$, i.e., polynomials with degree k, $p\in L^2(H^k(\Omega))$ and $D_tp\in L^2(H^{k-1}(\Omega))$. If w is chosen to be H^1 projection onto M of p, standard analysis and Lemma 4.1.1 or Theorem 4.3.1 imply that (see [36])*

$$
|||p-p_h|||_{\Omega_T}\le\alpha(h)+Ch^{k-1}.
$$

4.3.2. Incorporating Multiple Global Information

In applications, one often deals with multiple global fields that allow to represent the solution. We assume that p can be approximated by $G(p_1,...,p_N,t)$, where $p_i(x)$ ($i=1,2,...,N$) are pre-defined functions. Here, p, p_1, ..., p_N are scalar functions. More precisely, we assume there exists a function $G(\eta,t)\in L^\infty(H^2)\cap H^1(H^2)\cap$

$L^{\frac{2s}{s-2}}(W^{3,\frac{2s}{s-2}})$, $s > 2$, $\eta \in R^N$, such that

$$|||p - G(p_1, ..., p_N, t)|||_{\Omega_T} + ||D_t(p - G(p_1, ..., p_N, t))||_{L^2(L^2(\Omega))} \leq \delta, \tag{4.23}$$

where $p_i \in W^{1,s}(\Omega)(s > 2)$, $i = 1, 2, ..., N$. This is an extension of the previous approach. We would like to note that multiple global fields need to be used for stochastic differential equations [27]. Next, we define $M = span\{p_i | i = 1, 2, ..., N\}$, $p_1 = 1$ and follow Theorem 4.3.1 to approximate u. Let $\psi_{ij} = \phi_i^0 p_j$, then $\sum_i \phi_i^0 p_j = p_j$. Next, we study this method and provide convergence analysis. For this, we have the following theorem. This convergence theorem shows that multiscale finite element method converges independent of small scales in a more general setting when multiple global fields are used.

Theorem 4.3.2. *Under the assumption (4.23) and if $s > 4$, we have*

$$|||p - p_h|||_{\Omega_T} \leq \alpha(h) + \delta + Ch^{1-\frac{4}{s}}.$$

Proof. In the proof, after formulating stability estimate (4.24), we break the estimate into the estimates over each coarse patch. The estimation over each coarse patch is obtained using perturbation of G and regularities of basis functions and the global fields. By (4.4), we need to estimate the right hand sides of the following inequalities

$$|||p - w|||_{\Omega_T} \leq |||p - G(p_1, ..., p_N, t)|||_{\Omega_T} + |||G(p_1, ..., p_N, t) - c_{ij}\psi_{ij}|||_{\Omega_T}, \tag{4.24}$$

and

$$\begin{aligned} ||D_t(p - w)||_{L^2(L^2(\Omega))} \leq\ & ||D_t(p - G(p_1, ..., p_N, t))||_{L^2(L^2(\Omega))} \\ & + ||D_t(G(p_1, ..., p_N, t) - c_{ij}\psi_{ij})||_{L^2(L^2(\Omega))}, \end{aligned} \tag{4.25}$$

where $w = c_{ij}\psi_{ij}$ and c_{ij} is chosen such that the second term on the right hand of the

the above two inequalities is small. In each ω_i, we choose c_{ij} as

$$c_{i1} = G(\bar{p}_1^i, ..., \bar{p}_N^i, t) - \frac{\partial G}{\partial p_j}(\bar{p}_1^i, ..., \bar{p}_N^i, t)\bar{p}_j^i,$$

and

$$c_{ij} = \frac{\partial G}{\partial p_j}(\bar{p}_1^i, ..., \bar{p}_N^i, t), \quad j \geq 2,$$

where $\bar{p}_j^i$ is the average of p_j over ω_i. We note the following Taylor expansion in each ω_i,

$$G(p_1, ..., p_N, t) = G(\bar{p}_1^i, ..., \bar{p}_N^i, t) + \frac{\partial G}{\partial p_j}(\bar{p}_1^i, ..., \bar{p}_N^i, t)(p_j - \bar{p}_j^i) + R_i,$$

where R_i is the remainder, which can be written as

$$R_i = \sum_{k,j}(p_k - \bar{p}_k^i)(p_j - \bar{p}_j^i)\int_0^1 s\partial_{k,j}^2 G(p_1 + s(\bar{p}_1^i - p_1), ..., p_N + s(\bar{p}_N^i - p_N), t)ds,$$

where $\partial_{k,j}^2$ refers to second derivative with respect to p_k and p_j.

We first estimate (4.24). One can show that in each ω_i

$$\begin{aligned}
&|||G(p_1, ..., p_N, t) - c_{ij}p_j|||_{\omega_{iT}} \\
&\leq |||G(\bar{p}_1^i, ..., \bar{p}_N^i, t) + \frac{\partial G}{\partial p_j}(\bar{p}_1^i, ..., \bar{p}_N^i, t)(p_j - \bar{p}_j^i) - c_{ij}p_j|||_{\omega_{iT}} + |||R_i|||_{\omega_{iT}},
\end{aligned}$$

where $\omega_{iT} = \omega_i \times [0, T]$. The choice of c_{ij} implies that the first term on the right hand side is zero. We only need to estimate the second term $|||R_i|||_{\omega_{iT}}$. A straightforward computation (also see [4]) gives rise to

$$\begin{aligned}
\|R_i\|_{L^2(L^2(\omega_i))} &\leq \sum_{k,j} h^{2-\frac{4}{s}}|p_k|_{1,s,\omega_i}|p_j|_{1,s,\omega_i}|G|_{L^2(H^2)} \\
&\leq Ch^{2-\frac{4}{s}}\sum_j |p_j|_{1,s,\omega_i},
\end{aligned} \tag{4.26}$$

$$\begin{aligned} \|R_i\|_{L^\infty(L^2(\omega_i))} &\leq \sum_{k,j} h^{2-\frac{4}{s}} |p_k|_{1,s,\omega_i} |p_j|_{1,s,\omega_i} |G|_{L^\infty(H^2)} \\ &\leq C h^{2-\frac{4}{s}} \sum_j |p_j|_{1,s,\omega_i} \end{aligned} \tag{4.27}$$

and

$$\begin{aligned} |R_i|_{L^2(H^1(\omega_i))} \leq & \sum_{j,k,l} \|(p_j - \bar{p}_j^i)(p_k - \bar{p}_k^i) \nabla p_l \int_0^1 (1-s) s \partial^3_{k,j,l} G_s(.,t) ds\|_{L^2(L^2(\omega_i))} \\ &+ \sum_{j,k} \|((p_j - \bar{p}_j^i) \nabla p_j + (p_k - \bar{p}_k^i) \nabla p_k) \int_0^1 s \partial^2_{k,j} G_s(.,t) ds\|_{L^2(L^2(\omega_i))} \\ &\leq C \sum_{j,k,l} h^{2-\frac{4}{s}} |p_j|_{1,s,\omega_i} |p_k|_{1,s,\omega_i} (T |p_l|^s_{1,s,\Omega} + |G|^{\frac{2s}{s-2}}_{L^{\frac{2s}{s-2}}(W^{3,\frac{2s}{s-2}})})^{\frac{1}{2}} \\ &+ C \sum_j h^{1-\frac{2}{s}} |p_j|_{1,s,\omega_i} (T |p_j|^s_{1,s,\Omega} + |G|^{\frac{2s}{s-2}}_{L^{\frac{2s}{s-2}}(W^{2,\frac{2s}{s-2}})})^{\frac{1}{2}} \\ &\leq C \sum_{j,k} h^{2-\frac{4}{s}} |p_j|_{1,s,\omega_i} |p_k|_{1,s,\omega_i} + C \sum_j h^{1-\frac{2}{s}} |p_j|_{1,s,\omega_i} \\ &\leq C \sum_j h^{1-\frac{2}{s}} |p_j|_{1,s,\omega_i} \end{aligned} \tag{4.28}$$

where $\partial^3_{i,j,k}$ is partial triple partial derivative with respect to p_i, p_j, p_k, $G_s(.,t) = G(p_1 + s(\bar{p}_1^i - p_1), ..., p_N + s(\bar{p}_N^i - p_N), t)$ and Young's inequality is applied in the second step.

Notice that $\sum_i \phi_i^0 = 1$ and $|\nabla \phi_i^0| \le C\frac{1}{h}$, then it follows that

$$\begin{aligned}
&|G(p_1,...,p_N,t) - c_{ij}\psi_{ij}|^2_{L^2(H^1(\Omega))} \\
&= \int_\Omega |\nabla(G - c_{ij}\phi_i^0 p_j)|^2 dxdt \\
&= \int_\Omega |\nabla(\phi_i^0(G - c_{ij}p_j))|^2 dxdt \\
&\le C\int_\Omega |(G - c_{ij}p_j)\nabla\phi_i^0|^2 dxdt + C\int_\Omega |\phi_i^0 \nabla(G - c_{ij}p_j)|^2 dxdt \\
&\le C\frac{1}{h^2}\sum_i \|R_i\|^2_{L^2(L^2(\omega_i))} + C\sum_i |R_i|^2_{L^2(H^1(\omega_i))} \\
&\le Ch^{2-8/s}\sum_{i,j}|p_j|^2_{1,s,\omega_i} + Ch^{2-4/s}\sum_{i,j}|p_j|^2_{1,s,\omega_i} \\
&\le Ch^{2-8/s},
\end{aligned} \tag{4.29}$$

and

$$\begin{aligned}
|G(p_1,...,p_N,t) - c_{ij}\psi_{ij}|^2_{L^\infty(L^2(\Omega))} &\le C\sum_i \|R_i\|^2_{L^\infty(L^2(\omega_i))} \\
&\le Ch^{4-8/s}\sum_{i,j}|p_j|^2_{1,s,\omega_i} \\
&\le Ch^{4-8/s}.
\end{aligned}$$

It remains to estimate (4.25). We only need to estimate $\|D_t R_i\|_{L^2(L^2(\Omega))}$. It is easy to show

$$\begin{aligned}
\|D_t R_i\|_{L^2(L^2(\omega_i))} &\le \sum_{j,k}\|(p_j - \bar{p}_j^i)(p_k - \bar{p}_k^i)\int_0^1 s\partial_t \partial^2_{k,j} G_s(.,t)ds\|_{L^2(L^2(\omega_i))} \\
&\le C\sum_{j,k} h^{2-\frac{4}{s}}|p_j|_{1,s,\omega_i}|p_k|_{1,s,\omega_i}|\partial_t G|_{L^2(H^2)} \\
&\le Ch^{2-\frac{4}{s}}\sum_{j,k}|p_j|_{1,s,\omega_i}|p_k|_{1,s,\Omega} \\
&\le Ch^{2-\frac{4}{s}}\sum_j |p_j|_{1,s,\omega_i}
\end{aligned} \tag{4.30}$$

and consequently

$$\|D_t(G(p_1, ..., p_N, t) - c_{ij}\psi_{ij})\|^2_{L^2(L^2(\Omega))} \leq Ch^{4-8/s}.$$

Invoking (4.24) , (4.25) and (4.4), we complete the proof. $\square$

Comparing the convergence rate given by Theorem 4.2.1 and Theorem 4.3.2, the convergence rate by PUM is slightly less than that given by the Galerkin MsFEM using single global field in Theorem 4.2.1. However, if the patches ω_i have the uniform Poincaré property (as defined in [12]), then we can obtain the same convergence rate for these two methods. The result is given in the following corollary.

Corollary 4.3.3. *Provided that patches ω_i have the uniform Poincaré property and $s > 2$, then*

$$|||p - p_h|||_{\Omega_T} \leq \alpha(h) + \delta + Ch^{1-\frac{2}{s}}.$$

Proof. Let $A_{int} = \{i : \bar{\omega}_i \cap \partial\Omega = 0\}$ and $A_{bd} = \{i : \bar{\omega}_i \cap \partial\Omega \neq 0\}$. We choose

$$r_i(t) = \frac{1}{|\omega_i|}\int_{\omega_i}(G - c_{ij}(t)p_j)dx$$

for $i \in A_{int}$ and $r_i(t) = 0$ for $i \in A_{bd}$. we reset $c_{ij}(t)p_j$ in the proof of Theorem 4.3.2 to be $c_{ij}(t)p_j + r_i(t)$ in each patch ω_i.

Owing to

$$G(\bar{p}^i_1, ..., \bar{p}^i_N, t) + \frac{\partial G}{\partial p_j}(\bar{p}^i_1, ..., \bar{p}^i_N, t)(p_j - \bar{p}^i_j) - c_{ij}p_j = 0$$

in each ω_i, it follows immediately that

$$
\begin{aligned}
&\|G(p_1, ..., p_N, t) - c_{ij}p_j - r_i\|_{L^\infty(L^2(\omega_i))} \\
&\leq \|G(\bar{p}_1^i, ..., \bar{p}_N^i, t) + \frac{\partial G}{\partial p_j}(\bar{p}_1^i, ..., \bar{p}_N^i, t)(p_j - \bar{p}_j^i) - c_{ij}p_j - r_i\|_{L^\infty(L^2(\omega_i))} + \|R_i\|_{L^\infty(L^2(\omega_i))} \\
&= \|r_i\|_{L^\infty(L^2(\omega_i))} + \|R_i\|_{L^\infty(L^2(\omega_i))} \\
&= \|\frac{1}{|\omega_i|}\int_{\omega_i}(G - G(\bar{p}_1^i, \cdots, \bar{p}_N^i, t) - \partial_j G(\bar{p}_1^i, \cdots, \bar{p}_N^i, t)(p_j - \bar{p}_j^i))dx\|_{L^\infty(L^2(\omega_i))} \\
&+ \|R_i\|_{L^\infty(L^2(\omega_i))} \\
&= \|\frac{1}{|\omega_i|}\int_{\omega_i} R_i dx\|_{L^\infty(L^2(\omega_i))} + \|R_i\|_{L^\infty(L^2(\omega_i))} \\
&\leq Ch\|R_i\|_{L^\infty(L^2(\omega_i))} + \|R_i\|_{L^\infty(L^2(\omega_i))} \\
&\leq C\|R_i\|_{L^\infty(L^2(\omega_i))}.
\end{aligned}
\tag{4.31}
$$

Since patches ω_i have uniform Poincaré property by assumption in Corollary 4.3.3, Poincaré inequality implies that there exists a constant C independent of ω_i such that

$$
\begin{aligned}
\|G - c_{ij}p_j - r_i\|_{L^2(\omega_i)}(t) &\leq Ch\|G - c_{ij}p_j\|_{H^1(\omega_i)}(t) \\
\|G - c_{ij}p_j - r_i\|_{H^1(\omega_i)}(t) &\leq \|G - c_{ij}p_j\|_{H^1(\omega_i)}(t).
\end{aligned}
\tag{4.32}
$$

By (4.29) and (4.32), one can obtain that

$$
\begin{aligned}
&|G(p_1,...,p_N,t) - c_{ij}(t)\psi_{ij} - \phi_i^0 r_i|^2_{L^2(H^1(\Omega))} \\
&= \int_\Omega |\nabla(\phi_i^0((G - c_{ij}p_j - r_i))|^2 dxdt \\
&\le C\int_\Omega |(G - c_{ij}p_j - r_i)\nabla\phi_i^0|^2 dxdt + C\int_\Omega |\phi_i^0 \nabla(G - c_{ij}p_j - r_i)|^2 dxdt \\
&\le C\frac{1}{h^2}\sum_i h^2 \|G - c_{ij}p_j\|^2_{L^2(H^1(\omega_i))} + C\sum_i |G - c_{ij}p_j|^2_{L^2(H^1(\omega_i))} \\
&\le C(\sum_i (|R_i|_{L^2(H^1(\omega_i))})^2) \\
&\le C(\sum_i (\sum_j h^{1-\frac{2}{s}} |p_j|_{1,\omega_i})^2) \\
&\le Ch^{2-\frac{4}{s}}.
\end{aligned}
\tag{4.33}
$$

Utilizing (4.31) , (4.33) and following the proof of 4.3.2, we complete the proof. □

Remark 4.3.4. *If* $G(\eta,t) \in L^\infty(W^{2,\infty}) \cap H^1(H^2) \cap L^{\frac{2s}{s-2}}(W^{3,\frac{2s}{s-2}})$, $s > 2$, *then the proof of Theorem 4.3.2 implies that*

$$|||p - p_h|||_{\Omega_T} \le \alpha(h) + \delta + Ch^{1-\frac{2}{s}}.$$

If the number of global fields $p_1,\cdots,p_n$ is equal the dimension of Ω, then by using Taylor polynomials of Sobolev functions (see [17]) one can relax regularity of G and improve the convergence rate. For an example, we are going to consider the case of two global fields p_1, p_2 for $\Omega \subset R^2$, i.e., the local basis is a span of $\{1, p_1, p_2\}$. We define

$$
\begin{aligned}
T_i(x,y,t) =& G(y,t) + \nabla_x G(y,t)\cdot(x-y) \\
Q_i(x,t) =& \int_{B_i} T_i(x,y,t)\phi(y)dy \\
R_i(x,t) =& G(x,t) - Q_i(x,t),
\end{aligned}
\tag{4.34}
$$

where $x \in D_i \subset R^2$, $y \in D_i$, $B_i \subset\subset D_i$ is the ball centered at y_i with radius ρ and

$\phi(y)$ is the cut-off function supported in $\bar{B}_i$. Suppose D_i is star-shaped with respect to the ball B_i and $diam(D_i) \leq Cd$ uniformly for all i. If $G(.,t) \in H^2(D_i)$ for any t in its domain, then the following results hold [17],

$$\|R_i(.,t)\|_{0,\infty,D_i} \leq Cd|G(.,t)|_{2,D_i}, \tag{4.35}$$

$$|R_i(.,t)|_{k,D_i} \leq Cd^{2-k}|G(,.t)|_{2,D_i} \quad k = 0, 1. \tag{4.36}$$

Define a map $g : \omega_i \longrightarrow D_i$ via $x \longrightarrow (p_1(x), p_2(x))$, $H(x) = \frac{\partial(p_1,p_2)}{\partial(x_1,x_2)}$ and the Jacobian $J(x) = det\frac{\partial(p_1,p_2)}{\partial(x_1,x_2)}$. Assume that there exist constants C_1 , C_2 and C_3 such that

$$C_1 \leq |J(x)| \leq C_2 \tag{4.37}$$

and

$$\|H(x)\|_\infty \leq C_3 \tag{4.38}$$

uniformly hold for $x \in \Omega$, where

$$\|H(x)\|_\infty := \sup_{x\in\Omega}\left(\sup_{\xi\in R^2}\frac{(H(x)\xi,\xi)}{|\xi|^2}\right).$$

(4.37) means that $C_1h \leq d \leq C_2h$, where $diam(D_i) \approx d$ and $diam(\omega_i) \approx h$. One can use change of variables and the assumption of (4.37) and (4.38) to obtain the following inequalities:

$$\begin{aligned}
\|R_i(p_1(x), p_2(x), t)\|_{0,\infty,\omega_i} &\leq Ch\|G(.,t)\|_{2,D_i}, &(4.39)\\
\|R_i(p_1(x), p_2(x), t)\|_{0,\omega_i} &\leq Ch^2\|G(.,t)\|_{2,D_i}, &(4.40)\\
|R_i(p_1(x), p_2(x), t)|_{1,\omega_i} &\leq Ch\|G(.,t)\|_{2,D_i}. &(4.41)
\end{aligned}$$

The first two inequality (4.39) and (4.40) are easy to verify. We give a short proof for (4.41). In fact,

$$\begin{aligned} |R_i(p_1(x), p_2(x), t)|_{1,\omega_i} &\le (\inf_x J(x))^{-1/2} \|H(x)\|_\infty |R_i(., t)|_{1,D_i} \\ &\le Ch|G(., t)|_{2,D_i}, \end{aligned}$$

where we have used (4.37), (4.38) and (4.36) in the last step.

For the special case, we have the following theorem. We note that this theorem proves the convergence under weaker regularity assumptions for G.

Theorem 4.3.4. *Suppose that (4.37) and (4.38) hold. If $G \in H^1(H^2) \cap L^\infty(H^2)$, then*

$$|||p - p_h|||_{\Omega_T} \le \alpha(h) + \delta + Ch.$$

Proof. Let $\psi_{ij} = \phi_i^0 v_j$ ($i = 1, 2, .., m, j = 1, 2, 3$), where $v_1 = 1$, $v_2 = p_1$ and $v_3 = p_2$. Define $w = c_{ij}\psi_{ij}$, where

$$\begin{aligned} c_{i1} &= \int_{g^{-1}(B_i)} [G(p_1(y), p_2(y), t) - \partial_j G(p_1(y), p_2(y), t)p_j(y)]\phi(g^{-1}(y))dy, \\ c_{i,j+1} &= \int_{g^{-1}(B_i)} \partial_j G(p_1(y), p_2(y), t)\phi(g^{-1}(y))dy. \end{aligned} \tag{4.42}$$

By the proof Theorem 4.3.2, it is sufficient to estimate $\|R_i(p_1, p_2, t)\|_{L^2(L^2(\omega_i))}$, $\|R_i(p_1, p_2, t)\|_{L^\infty(L^2(\omega_i))}$, $\|R_i(p_1, p_2, t)\|_{L^2(H^1(\omega_i))}$ and $\|D_t R_i(p_1, p_2, t)\|_{L^2(L^2(\omega_i))}$. In fact,

$$\begin{aligned} \|R_i(p_1, p_2, t)\|_{L^2(L^2(\omega_i))} &\le Ch^2 \|G\|_{L^2(H^2(D_i))}, \\ \|R_i(p_1, p_2, t)\|_{L^\infty(L^2(\omega_i))} &\le Ch^2 \|G\|_{L^\infty(H^2(D_i))}, \\ \|R_i(p_1, p_2, t)\|_{L^2(H^1(\omega_i))} &\le Ch \|G\|_{L^2(H^2(D_i))}, \\ \|D_t R_i(p_1, p_2, t)\|_{L^2(L^2(\omega_i))} &\le Ch^2 \|D_t G\|_{L^2(H^2(D_i))}, \end{aligned} \tag{4.43}$$

where we have used (4.40) and (4.41). The rest of the proof is the same as that of Theorem 4.3.2. □

Remark 4.3.5. *The assumption on G in Theorem 4.3.4 is weaker than that of The-*

orem 4.3.2. At the same time, global fields p_1 and p_2 need to satisfy hypothesis (4.37) and (4.38). If $p_i \in W^{1,\infty}(\Omega)$, Theorem 4.3.4 and Theorem 4.3.2 simply imply the same convergence rate.

Remark 4.3.6. *Let p_i ($i = 1, 2, ..., d$ and $d = dim(\Omega)$) be the solution to*

$$\begin{aligned} div k(x)\nabla p_i = 0 \quad in \quad \Omega \\ p_i(x) = x_i \quad on \quad \partial\Omega, \end{aligned} \tag{4.44}$$

then $g(x)$ is an homeomorphism for $k(x) \in L^\infty(\Omega)$ [7]. Authors in [53] show $p(g^{-1}(p_1, p_2), t) \in L^2(H^2)$ and solve parabolic equations with continuous scales in (u_1, p_2) harmonic coordinate system(the meshes may be distorted). Theorem 4.3.4 implies the same convergence rate as that of harmonic coordinate method in [53] if we take $G(p_1, p_2, t) = p(g^{-1}(p_1, p_2), t)$.

4.3.3. Space and Time Dependent Global Information

In the previous subsection, the global fields $p_1, \cdots, p_N$ only depend on spatial variables. In this subsection, we consider a more general case that global fields depend on space and time, i.e., $p_i = p_i(x, t)$. We assume there exists a function $G \in H^3$ and such that

$$|||p - G(p_1, ..., p_N)|||_{\Omega_T} + ||D_t(p - G(p_1, ..., p_N))||_{L^2(L^2(\Omega))} \leq \delta, \tag{4.45}$$

where we assume

$$p_i \in H^1(L^2(\Omega)) \cap L^\infty(W^{1,s}(\Omega))(s > 2) \quad i = 1, 2, ..., m. \tag{4.46}$$

If we apply the PUM with global fields $p_i = p_i(x, t)$ ($i = 1, \cdots, N$), we have the following theorem for the convergence. This convergence theorem shows that multiscale finite element method converges independent of small scales in a more general setting when multiple global fields are used.

Theorem 4.3.5. *Under the assumptions (4.45) and (4.46), we have*

$$|||p - p_h|||_{\Omega_T} \leq \alpha(h) + \delta + C(h^{1-\frac{2}{s}} + h^{3-\frac{4}{s}}). \tag{4.47}$$

Proof. Because this proof is similar to that of Theorem 4.3.2, we only give a brief outline here. We know as before, it is sufficient to chose a proper $w : (0, T] \longrightarrow M$ and estimate $|||p-w|||_{\Omega_T}$ and $||D_t(p-w)||_{L^2(L^2(\Omega))}$. Set $w = c_{ij}\psi_{ij}$, where c_{ij} is chosen such that $c_{ij}\psi_{ij}$ is a good approximation of $G(p_1, ..., p_N)$. In each ω_i, we choose c_{ij} as

$$c_{i1} = G(\bar{p}_1^i, ..., \bar{p}_N^i) - \frac{\partial G}{\partial p_j}(\bar{p}_1^i, ..., \bar{p}_N^i)\bar{p}_j^i,$$

and

$$c_{ij} = \frac{\partial G}{\partial p_j}(\bar{p}_1^i, ..., \bar{p}_N^i), \quad j \geq 2,$$

where $\bar{p}_j^i$ is the average of p_j over ω_i. We note the following Taylor expansion in each ω_i,

$$G(p_1, ..., p_N) = G(\bar{p}_1^i, ..., \bar{p}_N^i) + \frac{\partial G}{\partial p_j}(\bar{p}_1^i, ..., \bar{p}_N^i)(p_j - \bar{p}_j^i) + R_i,$$

where R_i is the remainder, which can be given by

$$R_i = \sum_{k,j}(p_k - \bar{p}_k^i)(p_j - \bar{p}_j^i)\int_0^1 s\partial_{k,j}^2 G_s ds.$$

where we remind that $G_s = G(p_1 + s(\bar{p}_1^i - p_1), ..., p_N + s(\bar{p}_N^i - p_N))$. Triangle inequality implies that in each ω_i

$$\begin{aligned}
&|||G(p_1, ..., p_N) - c_{ij}p_j|||_{\omega_{iT}} \\
&\leq |||G(\bar{p}_1^i, ..., \bar{p}_N^i) + \frac{\partial G}{\partial p_j}(\bar{p}_1^i, ..., \bar{p}_N^i)(p_j - \bar{p}_j^i) - c_{ij}p_j|||_{\omega_{iT}} + |||R_i|||_{\omega_{iT}}.
\end{aligned} \tag{4.48}$$

The choice of c_{ij} implies that the first term on the right hand side is zero. We only need to estimate the second term. A direct computation gives

$$\begin{aligned}\|R_i\|_{L^2(L^2(\omega_i))} &\leq Ch^{3-\frac{4}{s}}|G|_2\sum_{j,k}|p_j|_{L^\infty(W^{1,s}(\omega_i))}|p_k|_{L^\infty(W^{1,s}(\omega_i))}\\ &\leq Ch^{3-\frac{4}{s}}\sum_{j}|p_j|_{L^\infty(W^{1,s}(\omega_i))},\end{aligned}$$

$$\|R_i\|_{L^\infty(L^2(\omega_i))} \leq Ch^{3-\frac{4}{s}}\sum_{j}|p_j|_{L^\infty(W^{1,s}(\omega_i))}$$

$$|R_i|_{L^2(H^1(\omega_i))} \leq Ch^{1-\frac{2}{s}}\sum_{l}|p_l|_{L^2(H^1(\omega_i))},$$

and

$$\begin{aligned}|R_i|_{L^2(H^1(\omega_i))} &\leq \sum_{j,k,l}\|(p_j-\bar{p}_j^i)(p_k-\bar{p}_k^i)\nabla p_l\int_0^1(1-s)s\partial^3_{k,j,l}G_s ds\|_{L^2(L^2(\omega_i))}\\ &+\sum_{j,k}\|((p_j-\bar{p}_j^i)\nabla p_j+(p_k-\bar{p}_k^i)\nabla p_k)\int_0^1 s\partial^2_{k,j}G_s ds\|_{L^2(L^2(\omega_i))}\\ &\leq C\sum_{j,k,l}h^{2-\frac{4}{s}}|p_j|_{L^\infty(W^{1,s}(\omega_i))}|p_k|_{L^\infty(W^{1,s}(\omega_i))}|p_l|_{L^2(H^1(\omega_i))} \qquad (4.49)\\ &+C\sum_{j}h^{1-\frac{2}{s}}|p_j|_{L^\infty(W^{1,s}(\omega_i))}|p_j|_{L^2(H^1(\omega_i))}\\ &\leq Ch^{1-\frac{2}{s}}\sum_{j}|p_j|_{L^2(H^1(\omega_i))}.\end{aligned}$$

Consequently, we can obtain

$$|G(p_1,...,p_N)-c_{ij}\psi_{ij}|^2_{L^2(H^1(\Omega))} \leq Ch^{2-4/s},$$

and

$$|G(u_1,...,p_N)-c_{ij}\psi_{ij}|^2_{L^\infty(L^2(\Omega))} \leq Ch^{6-8/s}.$$

If $p_i \in L^\infty(W^{1,s}(\Omega)) \cap H^1(H^1(\Omega))$, one can show that

$$\begin{aligned}
\|D_t R_i\|_{L^2(L^2(\omega_i))} &\leq \sum_{j,k} \|D_t((p_j - \bar{p}^i_k)(p_j - \bar{p}^i_k)) \int_0^1 s\partial^2_{j,k} G_s ds\|_{L^2(L^2(\omega_i))} \\
&+ \sum_{j,k,l} \|(p_j - \bar{p}^i_j)(p_k - \bar{p}^i_k) D_t p_l \int_0^1 s(1-s)\partial^3_{j,k,l} G_s ds\|_{L^2(L^2(\omega_i))} \\
&+ \sum_{j,k,l} \|(u_j - \bar{p}^i_j)(p_k - \bar{p}^i_k) D_t \bar{p}^i_l \int_0^1 s^2 \partial^3_{j,k,l} G_s ds\|_{L^2(L^2(\omega_i))} \\
&\leq Ch^{2-\frac{2}{s}} |G|_2 \sum_{j,k} |p_j|_{L^\infty(W^{1,s}(\omega_i))} |D_t p_k|_{L^2(H^1(\omega_i))} \\
&+ Ch^{2-\frac{4}{s}} |G|_3 \sum_{j,k,l} |p_j|_{L^\infty(W^{1,s}(\omega_i))} |p_k|_{L^\infty(W^{1,s}(\omega_i))} |D_t p_l|_{L^2(L^2(\omega_i))} \\
&\leq Ch^{2-\frac{4}{s}} \sum_l \|D_t p_l\|_{L^2(H^1(\omega_i))}.
\end{aligned} \tag{4.50}$$

Consequently, it follows

$$\|D_t(G(p_1, ..., p_N) - c_{ij}\psi_{ij})\|^2_{L^2(L^2(\Omega))} \leq Ch^{4-8/s}.$$

The rest of the proof is the same as that of Theorem 4.3.2. $\square$

Remark 4.3.7. *Let $p_i(x,t)$ ($i = 1, 2, ..., d$ and $d = dim(\Omega)$) is the solution of the following equation*

$$\begin{cases}
D_t p_i - div k(x,t)\nabla p_i &= 0 \quad in \quad \Omega_T \\
p_i &= x_i \quad on [0,T) \times \partial\Omega \\
div k(x,0)\nabla p_i(x,0) &= 0 \quad in \quad \Omega.
\end{cases}$$

Define a map $g : (x,t) \longrightarrow (p_1(x,t), \cdots, p_n(x,t))$, authors in [52] showed $p(g^{-1}(p_1, \cdots, p_n), t) \in L^2(H^2) \cap H^1(L^2)$ (if f and initial values are smooth enough) and solved the parabolic equations in terms of $(p_1, \cdots, p_n)$ system. Let finite dimension space be defined by

$$\hat{M} = \{\varphi(g(x,t)) : \varphi \in X_h\},$$

where X_h is some standard finite element space like weighted extended B-splines space. If $p_h \in \hat{M}$, then it is shown in [52] that

$$|||p - p_h|||_{\Omega_T} \leq Ch.$$

Let

$$\|D_t(p(t) - \xi_j^p(t))\|_{-1,\omega_j}^2 \leq C\tilde{\epsilon}_3(t,j)^2 \tag{4.51}$$

According to Lemma 4.1.1, we can adapt the proof of Theorem 4.3.1 and easily get the following result.

Theorem 4.3.6. *Let p and p_h be the solutions to (4.1) and (4.2) respectively. If (4.18) and (4.51) hold, then*

$$|||p - p_h|||_{\Omega_T}^2 \leq C\alpha(h) + C\int_0^T (\epsilon_1(t,j)^2 + \frac{\epsilon_1^2(t,j)}{h_j^2} + \epsilon_2(t,j)^2 + \tilde{\epsilon}_3(t,j)^2)dt. \tag{4.52}$$

One can apply Theorem 4.3.6 to parabolic equations as we did before and obtain the convergence rate under a weaker assumption (4.51) instead of (4.18).

4.3.4. Time Discretization of PUM for Parabolic Equations

In Section 4.3, we have considered the semi-discretization of the parabolic equation (4.2) using Galerkin MsFEM and PUM. In practice, one needs to discretize the temporal variables also. In this section, we present time discretization of multiscale parabolic equation when multiple global fields are used to construct basis functions. To fully discretize systems, we introduce the following notation. If $v_h : \{t_m\}_0^M \longrightarrow X$, where M is a positive integer. Let $\Delta t = \frac{T}{M}$ and X be normed space with norm $\|.\|_X$

(similar definition for X to be semi-norm space), then

$$\begin{aligned}
\|v_h\|_{L^\infty_\Delta(X)} &= \max_{0\le m\le M} \|v_h^m\|_X \\
\|v_h\|^2_{L^2_\Delta(X)} &= \sum_{m=0}^{M} \|v_h^m\|_X^2 \Delta t \\
\|v_h\|^2_{\tilde{L}^2_\Delta(X)} &= \sum_{m=0}^{M-1} \|v_h^{m+1}\|_X^2 \Delta t.
\end{aligned}$$

We use the backward Euler scheme to discretize the time, i.e.,

$$(D_t p_h^m, v) + (k^{m+1}\nabla p_h^{m+1}, \nabla v) = (f^{m+1}, v) \quad v \in M, \tag{4.53}$$

where $D_t p_h^m = \frac{p_h^{m+1} - p_h^m}{\Delta t}$, $k^{m+1} = k(x, t_{m+1})$ and $f^{m+1} = f(x, t_{m+1})$ if f is sufficiently smooth with respect to temporal variable. We assume that p^m can be approximated by $\xi_j(p^m)$ on each patch w_j and the following estimates in each patch ω_i are satisfied:

$$\|p^m - \xi_j(p^m)\|^2_{0,\omega_j} \le C\epsilon_1^2(m,j), \tag{4.54}$$

$$|p^m - \xi_j(p^m)|^2_{1,\omega_j} \le C\epsilon_2^2(m,j), \tag{4.55}$$

$$\|D_t(p^m - \xi_j(p^m))\|^2_{0,\omega_j} \le C\epsilon_3^2(m,j). \tag{4.56}$$

Let us define

$$|||p - p_h|||_{\Omega_T,\Delta} = \|p - p_h\|_{L^\infty_\Delta(L^2(\Omega))} + |p - p_h|_{\tilde{L}^2_\Delta(H^1(\Omega))}.$$

In this subsection, we always assume p_h is the solution of (4.53), unless otherwise is stated.

Based on the backward Euler scheme in (4.53) and PUM in section 4.3.1, the following theorem follows. This theorem estimates spatio-temporal convergence rate through the approximation estimates on the patches.

Theorem 4.3.7. *Let p and p_h^m be the solutions to (4.1) and (4.53) respectively. If*

(4.54) holds and $supp_t D_{tt} p \le C$*, then*

$$|||p - p_h|||^2_{\Omega_T,\Delta} \le C(\epsilon_1(m,j)^2\Delta t + \frac{\epsilon_1(m,j)^2\Delta t}{h_j^2} + \epsilon_2(m,j)^2\Delta t + \epsilon_3(m,j)^2\Delta t) + \alpha(h) + C\Delta t. \tag{4.57}$$

Proof. Suppose w is an arbitrary map from $[0,t]$ into M, and $\xi = p_h - w$, $\eta = p - w$ and consistency error $R^{m+1} = \frac{p_h^{m+1} - p_h^m}{\Delta t} - D_t p(t_{m+1})$, then we obtain that

$$(D_t\xi^m, v) + (k^{m+1}\nabla\xi^{m+1}, \nabla v) = (k^{m+1}\nabla\eta^{m+1}, \nabla v) + (D_t\eta^m, v) + (R^{m+1}, v), \quad v \in M.$$

If we choose the test function $v = \xi^{m+1}$ and notice that

$$(D_t\xi^m, \xi^{m+1}) = \frac{1}{2\Delta t}(\|\xi^{m+1}\|^2_{0,\Omega} - \|\xi^m\|^2_{0,\Omega} + \|\xi^{m+1} - \xi^m\|^2_{0,\Omega}),$$

then by applying coerciveness and Schwartz inequality, we have

$$\begin{aligned}\frac{1}{2\Delta t}(\|\xi^{m+1}\|^2_{0,\Omega} - \|\xi^m\|^2_{0,\Omega}) + C|\xi^{m+1}|^2_{1,\Omega} &\le \frac{C}{2\gamma}|\eta^{m+1}|^2_{1,\Omega} + \frac{\gamma}{2}|\xi^{m+1}|^2_{1,\Omega} \\ &+ \frac{1}{2}(\|\xi^{m+1}\|^2_{0,\Omega} + \|D_t\eta^m\|^2_{0,\Omega}) + \frac{1}{2}(\|R^{m+1}\|^2_{0,\Omega} + \|\xi^{m+1}\|^2_{0,\Omega}).\end{aligned}$$

With a proper choice of γ and summing m from 0 to $q-1$ ($q \le M$) we get

$$\begin{aligned}&\|\xi^q\|^2_{0,\Omega} - \|\xi^0\|^2_{0,\Omega} + \sum_{m=0}^{q-1}|\xi^{m+1}|^2_{1,\Omega}\Delta t \\ &\le C\Delta t\sum_{m=0}^{q-1}\|\xi^{m+1}\|^2_{0,\Omega} + C(\|D_t\eta\|^2_{L^2_\Delta(L^2(\Omega))} + |\eta|^2_{\tilde{L}^2_\Delta(H^1(\Omega))} + \|R\|^2_{\tilde{L}^2_\Delta(L^2(\Omega))}).\end{aligned} \tag{4.58}$$

When Δt is sufficiently small, the discrete Gronwall's inequality (i.e., Remark 2.2.7) implies that

$$\begin{aligned}&\|\xi\|^2_{L^\infty_\Delta(L^2(\Omega))} + |\xi|^2_{\tilde{L}^2_\Delta(H^1(\Omega))} \\ &\le C(\|D_t\eta\|^2_{L^2_\Delta(L^2(\Omega))} + |\eta|^2_{\tilde{L}^2_\Delta(H^1(\Omega))} + \|R\|^2_{\tilde{L}^2_\Delta(L^2(\Omega))} + \|\xi(0)\|^2_{0,\Omega}).\end{aligned} \tag{4.59}$$

Owing to $p - p_h = \eta - \xi$, triangle inequality gives rise to

$$|||p - p_h|||_{\Omega_T,\Delta} \leq C(|||p - w|||_{\Omega_T,\Delta} + \|D_t(p - w)\|_{L^2_\Delta(L^2(\Omega))} \\ + \|R\|^2_{\tilde{L}^2_\Delta(L^2(\Omega))} + \|p_{0,h} - w(0)\|_{0,\Omega}). \tag{4.60}$$

Because

$$\begin{aligned} R^{m+1} &= \frac{p_h^{m+1} - p_h^m}{\Delta t} - D_t p(t_{m+1}) \\ &= \frac{1}{\Delta t}\int_{t_n}^{t_{n+1}} (D_t p_h(t) - D_t p(t))dt - \frac{1}{\Delta t}\int_{t_n}^{t_{n+1}} (t - t_n) D_{tt} p(t) dt, \end{aligned}$$

Peano Kernel Theorem (e.g., [36]) implies that

$$\|R\|^2_{\tilde{L}^2_\Delta(L^2(\Omega))} \leq C \sup_t \|D_{tt} u\|_{0,\Omega} \Delta t.$$

Taking $w^m = \sum_j \phi_j^0 \xi_j(p^m)$ and following the proof of Theorem 4.3.1, we easily complete the proof. □

Remark 4.3.8. *If patches w_j have the uniform Poincaré property, one can simplify (4.57) to be*

$$|||p - p_h|||^2_{\Omega_T,\Delta} \leq C(h_j^2 \epsilon_2(m,j)^2 \Delta t + \epsilon_2(m,j)^2 \Delta t + \epsilon_3(m,j)^2 \Delta t) \\ + \alpha(h) + C\Delta t. \tag{4.61}$$

By the proof the Theorem 4.3.7, inequality (4.57) can be reduced to

$$|||p - p_h|||^2_{\Omega_T,\Delta} \leq \sup_{t^m} \sum_j \epsilon_1(t^m, j) + C(\frac{\epsilon_1(m,j)^2 \Delta t}{h_j^2} + \epsilon_2(m,j)^2 \Delta t + \epsilon_3(m,j)^2 \Delta t) \\ + \alpha(h) + C\Delta t. \tag{4.62}$$

Remark 4.3.9. *Let the finite space $M \in P_k$, $p \in L^2(H^k(\Omega))$ and $D_{tt} p \in L^\infty(L^2(\Omega))$. If w is chosen to be H^1 projection onto M of p, the proof of Theorem 4.3.7 imply that*

$$|||p - p_h|||_{\Omega_T,\Delta} \leq \alpha(h) + C(h^{k-1} + \Delta t).$$

We assume there exists a function $G(\eta, t) \in L^\infty(H^2) \cap H^1(H^2) \cap L^{\frac{2s}{s-2}}(W^{3,\frac{2s}{s-2}})$, $(s > 2)$ such that

$$|||p - G(p_1, ..., p_N, t)|||_{\Omega_T,\Delta} + ||D_t(p - G(p_1, ..., p_N, t))||_{L^2_\Delta(L^2(\Omega))} \le \delta, \tag{4.63}$$

where $p_i = p_i(x) \in W^{1,s}(\Omega)(s > 2)$, $i = 1, 2, ..., N$. We utilize PUM and apply Theorem 4.3.7 and repeat the same procedure as that in the proof of Theorem 4.3.2, and we have the following result immediately.

Theorem 4.3.8. *Under assumption (4.63) and $D_{tt}p \in L^\infty(L^2(\Omega))$, if $s > 4$, then*

$$|||p - p_h|||_{\Omega_T,\Delta} \le \alpha(h) + \delta + C(h^{1-\frac{4}{s}} + \Delta t).$$

Following the similar analysis presented in the previous section, one can obtain the corollary as following.

Corollary 4.3.9. *Provided that patches ω_i have the uniform Poincaré property and $s > 2$, then*

$$|||p - p_h|||_{\Omega_T,\Delta} \le \alpha(h) + \delta + C(h^{1-\frac{2}{s}} + \Delta t).$$

As a discrete analogue of Theorem 4.3.4, we have the theorem as following.

Theorem 4.3.10. *Suppose that (4.37) and (4.38) hold. If $D_{tt}p \in L^\infty(L^2(\Omega))$ and $G \in H^1_\Delta(H^2) \cap L^\infty_\Delta(H^2)$, then*

$$|||p - p_h|||_{\Omega_T,\Delta} \le \alpha(h) + \delta + C(h + \Delta t).$$

The proof is similar to that of Theorem 4.3.4, and we only need to switch the time integral to the discrete time summation.

When $D_{tt}p \notin L^\infty(L^2(\Omega))$, standard time discretization may not good enough. For this case, we suggest to apply some weak implicit scheme to discretize time.

Owhadi and Zhang in [52] proposed a weak implicit Euler method. We may use this scheme for our PUM method. The idea is the following. Let $\psi_{ij} = \phi_i^0(x)p_j$. If

$$w_h = c_{ij}(t)\psi_{ij},$$

we define

$$w_h^n = c_{ij}(t_n)\psi_{ij}.$$

The weak implicit scheme is to seek v_h^n such that

$$\begin{aligned}(v_h^{n+1}(t_{n+1}), \psi_{ij}(t_{n+1})) = (v_h^n(t_n), \psi_{ij}(t_n)) + \int_{t_n}^{t_{n+1}} &[(v_h^{n+1}, D_t\psi_{ij}(t)) \\ &- (k\nabla v_h^{n+1}(t), \nabla\psi_{ij}(t)) - (f(t), \psi_{ij}(t))]dt\end{aligned} \tag{4.64}$$

for all shape functions ψ_{ij}. Let p_h and v_h be the time continuous solution of (4.2) and the solution to (4.64), respectively. Then one can follow the techniques of [52] and derive the following convergence rate for space dependent global fields.

$$|||p_h - v_h|||_{\Omega_T,\Delta} \le C\Delta t.$$

If $p = G(p_1, p_2, \cdots, p_N, t)$, N is the dimension of the domain Ω and $p_i = p_i(x)$ is the solution of (4.44). For the space-time dependent fields, the convergence rate is given by

$$|||p_h - v_h|||_{\Omega_T,\Delta} \le C\frac{\Delta t}{h}$$

if $p = G(p_1, p_2, \cdots, p_N)$, N is the dimension of the domain Ω, and $p_i = p_i(x, t)$ solve the parabolic equation with some prescribed boundary and initial conditions as defined in [52].

4.4. Mixed MsFEM for Parabolic Equations Using Global Information

In this section we investigate the mixed MsFEM using multiple global information to the model parabolic equation (4.1) with homogeneous Dirichlet boundary condition. We note that parabolic equations arise in two-phase flow simulations when compressibility is present. For convenience, we dropped $\lambda(x)$ from the equations since it does not change the analysis. One can assume $k(x) = \lambda(x)K(x)$, where $K(x)$ is a heterogeneous field and $\lambda(x)$ is a smooth field. Let $u = -k(x)\nabla p$, then

$$\begin{aligned} D_t p + divu &= f \\ (k(x))^{-1}u + \nabla p &= 0. \end{aligned} \tag{4.65}$$

We will use the following assumption for the parabolic equation.

Assumption A1p. There exist functions $u_1, \cdots, u_N$ and sufficiently smooth $A_1(t,x), \cdots, A_N(t,x)$ such that

$$u(t,x) = A_i(t,x)u_i,$$

where $u_i = k\nabla p_i$ and p_i solves $div(k(x)\nabla p_i) = 0$ in Ω with appropriate boundary conditions.

For our analysis, we assume, as before, $A_i(t,x) \in L^2(0,T;W^{1,\xi}(\Omega))(\xi > 2)$ and $u_i = k(x)\nabla p_i \in L^\eta(\Omega)$ $(\frac{1}{2} = \frac{1}{\xi} + \frac{1}{\eta})$, $i = 1, ..., N$.

Remark 4.4.1. *Let $u_i = k(x)\nabla p_i$ $(i = 1, \cdots, d$ and $d = dim(\Omega))$ be defined in (3.118), then Owhadi and Zhang in [52] show that $p(t,x) = p(t,p_1,p_2) \in L^2(0,T;W^{2,s})$ $(s > 2)$. Consequently, $u(t,x) = k(x)\nabla p = \frac{\partial p}{\partial p_i}k\nabla p_i := A_i(t,x)u_i$, where $A_i(t,x) = \frac{\partial p}{\partial p_i} \in L^2(0,T;W^{1,s}(\Omega))$.*

The mixed formulation associated to (4.1) is to find $\{u,p\} : [0,T] \longrightarrow H(div,\Omega)\times$

$L^2(\Omega)$ such that

$$\left\{\begin{array}{rcl}(D_t p, q) + (divu, q) &=& (f, q) \quad \forall q \in L^2(\Omega) \\ (k^{-1}u, v) - (divv, p) &=& 0 \quad \forall v \in H(div, \Omega) \\ p(0) &=& p_0.\end{array}\right. \tag{4.66}$$

Let finite dimensional space V_h and Q_h be defined as in Section 3.3.3, then $V_h \times Q_h \subset H(div, \Omega) \times L^2(\Omega)$ and the space-discrete mixed formulation is to find $\{u_h, p_h\}$: $[0, T] \longrightarrow V_h \times Q_h$ such that

$$\left\{\begin{array}{rcl}(D_t p_h, q_h) + (divu_h, q_h) &=& (f, q_h) \quad \forall q_h \in Q_h \\ (k^{-1}u_h, v_h) - (divv_h, p_h) &=& 0 \quad \forall v_h \in V_h \\ p_h(0) &=& p_{0,h},\end{array}\right. \tag{4.67}$$

where $p_{0,h}$ is the L^2 projection of p_0 onto Q_h. Because (3.132) holds, the problem (4.67) is well-posed. This problem can be rewritten in matrix form

$$\begin{aligned} MD_t P + BU &= F \\ B^t P - LU &= 0 \end{aligned} \tag{4.68}$$

with $P(0)$ given, where M and L are symmetric positive and definite. After eliminating U, (4.68) is a linear system ODEs for P,

$$MD_t P + BL^{-1}B^t P = F.$$

Since M is positive and definite, this system has a unique solution.

We define

$$\|u\|^2_{L^2_k(\Omega)} = \int_\Omega u^t \cdot k^{-1}(x) u dx$$

and

$$\|u\|^2_{L^2(0,T;L^2_k(\Omega))} = \int_0^T \int_\Omega u^t \cdot k^{-1}(x) u dx ds.$$

Let $\Pi_h : H(div) \longrightarrow V_h$ be the interpolation operator defined as in Section 3.3.3 and $P_{Q_h} : L^2(\Omega) \longrightarrow Q_h$ be the L^2 projection onto Q_h.

From (4.66) and (4.67), we have

$$\begin{aligned} (D_t(p-p_h), q_h) + (div(u-u_h), q_h) = 0 \quad \forall q_h \in Q_h \\ (k^{-1}(u-u_h), v_h) - (div v_h, p-p_h) = 0 \quad \forall v_h \in V_h. \end{aligned} \tag{4.69}$$

Taking $v_h = \Pi_h u - u_h$ and $q_h = P_{Q_h} p - p_h$, we have

$$\begin{aligned} (D_t(p-p_h), P_{Q_h}p - p_h) + (div(u-u_h), P_{Q_h}p - p_h) = 0 \\ (k^{-1}(u-u_h), \Pi_h u - u_h) - (div(\Pi_h u - u_h), p - p_h) = 0. \end{aligned} \tag{4.70}$$

Rewriting $p - p_h = p - P_{Q_h}p + P_{Q_h}p - p_h$ and $u - u_h = u - \Pi_h u + \Pi_h u - u_h$ in (4.70) and making summation of the two equalities, we obtain

$$\begin{aligned} &(D_t(P_{Q_h}p - p_h), P_{Q_h}p - p_h) + (k^{-1}(\Pi_h u - u_h), \Pi_h u - u_h) \\ &= -(D_t(p - P_{Q_h}p), P_{Q_h}p - p_h) - (k^{-1}(u - \Pi_h u), \Pi_h u - u_h) \\ &+ (div(\Pi_h u - u_h), p - P_{Q_h}p) - (div(u - \Pi_h u), P_{Q_h}p - p_h). \end{aligned} \tag{4.71}$$

Since P_{Q_h} is $L^2(\Omega)$ projection onto Q_h, P_{Q_h} commutes with the time derivative operator D_t. Consequently, the first and third term of the right hand in (4.71) are vanished. By Lemma 3.3.9, the fourth term of the right hand in (4.71) is also vanished. Consequently, (4.71) becomes

$$\begin{aligned} &(D_t(P_{Q_h}p - p_h), P_{Q_h}p - p_h) + (k^{-1}(\Pi_h u - u_h), \Pi_h u - u_h) \\ &= -(k^{-1}(u - \Pi_h u), \Pi_h u - u_h). \end{aligned} \tag{4.72}$$

Schwarz inequality and Young's inequality give rise to

$$\frac{1}{2} D_t \|P_{Q_h}p - p_h\|_{0,\Omega}^2 + 2\|\Pi_h u - u_h\|_{L_k^2(\Omega)}^2 \leq \lambda \|\Pi_h u - u_h\|_{L_k^2(\Omega)}^2 + \frac{1}{4\lambda}\|u - \Pi_h u\|_{L_k^2(\Omega)}^2. \tag{4.73}$$

Integrating with respect with time and applying Gronwall's inequality and after choosing proper value for λ, we have

$$\begin{aligned}
&\|P_{Q_h}p - p_h\|^2_{C^0(0,T;L^2(\Omega))} + \|\Pi_h u - u_h\|^2_{L^2(0,T;L^2_k(\Omega))} \\
&\quad \le C(\|P_{Q_h}p(0) - p_{0,h}\|^2_{0,\Omega} + \|u - \Pi_h u\|^2_{L^2(0,T;L^2_k(\Omega))}).
\end{aligned} \tag{4.74}$$

Invoking the triangle inequality, we have

$$\begin{aligned}
&\|p - p_h\|^2_{C^0(0,T;L^2(\Omega))} + \|u - u_h\|^2_{L^2(0,T;L^2_k(\Omega))} \\
&\le C(\|P_{Q_h}p(0) - p_{0,h}\|^2_{0,\Omega} + \|u - \Pi_h u\|^2_{L^2(0,T;L^2_k(\Omega))}) + \|p - P_{Q_h}p\|^2_{C^0(0,T;L^2(\Omega))}.
\end{aligned} \tag{4.75}$$

Hence, we obtain the following lemma.

Lemma 4.4.1. *Let $\{u, p\}$ and $\{u_h, p_h\}$ be the solution of (4.66) and (4.67) respectively. Under Assumption A1p and the definition of V_h in Section 3.3.3, (4.75) holds.*

Utilizing Lemma 4.4.1 and the proof of Theorem 3.3.12, we can derive the convergence result.

Theorem 4.4.2. *Let $\{u, p\}$ and $\{u_h, p_h\}$ be the solution of (4.66) and (4.67) respectively. If $\alpha + \beta_1 - \beta_2 - 1 > 0$ then*

$$\|p - p_h\|_{C^0(0,T;L^2(\Omega))} + \|u - u_h\|_{L^2(0,T;L^2_k(\Omega))} \le Ch^{\min\{\alpha+\beta_1-\beta_2-1,1\}},$$

where $\alpha = 1 - \frac{2}{\xi}$ and ξ is from Assumption A1p, and β_i $(i = 1, 2)$ are defined in Assumption A2 described in Section 3.3.3.

Proof. Owing to the fact that P_{Q_h} is the $L^2(\Omega)$ projection onto Q_h,

$$\|p - P_{Q_h}p\|_{C^0(0,T;L^2(\Omega))} \le Ch|p|_{C^0(0,T;H^1(\Omega))}, \tag{4.76}$$

this estimates the first and the third term of right hand side in (4.75). Next we estimate the term $\|u - \Pi_h u\|^2_{L^2(0,T;L^2_k(\Omega))}$. Define

$$A^K_{ij}(t) = \int_{e_j} A_i(t, s)u_i \cdot n ds$$

in each element K. Because $k^{-1}(x)$ is bounded, we have in each element K

$$
\begin{aligned}
&\|u-\Pi_h u\|^2_{L^2(0,T;L^2_k(K))} = \\
&\int_0^T \int_K (A_i(t,x)\beta_{ij}^K - A_{ij}^K(t))\psi_{ij}^K \cdot k^{-1}(A_i(t,x)\beta_{ij}^K - A_{ij}^K(t))\psi_{ij}^K dxdt \\
&\leq C\int_0^T \int_K ((A_i(t,x)\beta_{ij}^K - A_{ij}^K(t))\psi_{ij}^K)^2 dxdt \\
&= C\|(A_i(t,x)\beta_{ij}^K - A_{ij}^K(t))\psi_{ij}^K\|^2_{L^2(0,T;L^2(K))} \\
&\leq C\|(A_i(t,x) - \bar{A}_i^j(t))\beta_{ij}^K\psi_{ij}^K\|^2_{L^2(0,T;L^2(K))} \\
&+ C\|(\bar{A}_i^j(t)\beta_{ij}^K - A_{ij}^K(t))\psi_{ij}^K\|^2_{L^2(0,T;L^2(K))} \\
&\leq Ch^{2(\alpha+\beta_1)}\|\psi_{ij}^K\|^2_{0,K}.
\end{aligned}
\tag{4.77}
$$

In the last step, we used that facts that $A_i \in L^2(0,T;W^{1,\xi})$, *Assumption A2* and proof of Theorem 3.3.12 (see (3.141)). After making summation over all K for (4.77), we have

$$
\|u-\Pi_h u\|_{L^2(0,T;L^2_k(\Omega))} \leq Ch^{(\alpha+\beta_1-\beta_2-1)}. \tag{4.78}
$$

Now, the proof can be completed taking into account (4.76) and (4.78). □

4.5. Numerical Results

In this section, we present a few numerical results to demonstrate the importance of incorporating global information for a parabolic equation. We will consider multiscale finite element methods with limited global information presented in Section 4.2 and restrict ourselves to nonlinear parabolic equations (Richards equations)

$$
D_t\theta(p) - \nabla\cdot(k_r(p)k(x)\nabla p) = 0 \tag{4.79}
$$

with some prescribed boundary and initial conditions. In this case, $k_r(p)$ acts as a smooth function, consequently, does not modify the small scale information which

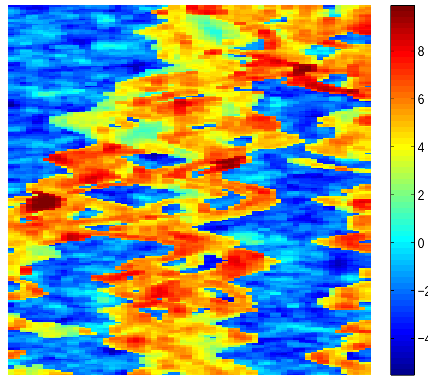

Fig. 4.1. Log of permeability field of layer 40 of SPE 10 SPE comparative project [24]

is contained in $k(x)$. In our numerical tests, we will use a channelized permeability field $k(x)$ which induces strong non-local effects. This type of heterogeneities is presented in a benchmark test of the SPE comparative project [24] (upper Ness layers). These permeability fields are highly heterogeneous, channelized, and difficult to upscale. In Figure 4.1, we plot a log of this field. As it can be observed, the irregular channels introduces strong non-locality across the entire domain. For these types of heterogeneities, usually local approaches fail to give an accurate results.

In our numerical tests, we compare the solution at several time instances. The following boundary conditions are prescribed: $p = 1$ along the $x = 0$ edge and $p = 0$ along the $x = 1$ edge. We assume initially $p_0 = 0$ in the entire region and compare the following methods. In our numerical examples, we take $\theta(p) = \exp(0.01p)$ and $k_r(p) = \exp(-p)$. Our first choice is standard MsFEM with local basis functions where linear boundary conditions are imposed for these basis functions. For the second approach, MsFEM with oversampling [38] is chosen. Here, larger regions (larger than the target coarse grid block) are used to construct auxiliary basis functions. The local nodal basis functions are constructed using linear combinations of these auxiliary functions. The size of the global domain influences the convergence of the method and, in particular, larger regions provide better accuracy. For general heterogeneities, it is difficult to determine optimal oversampling domain size and this is the reason of using global approaches in subsurface applications. In our numerical comparisons, oversampling is used with one extra coarse block extension to calculate the basis functions. Finally, we use MsFEM with limited global information presented in Section 4.2 where steady state solution is used to generate basis functions. Note that the construction of the basis functions is one time overhead that involves the solution of elliptic partial differential equation. Using these basis functions, the fine-scale equation can be solved repeatedly for various boundary conditions and right hand sides. For solving (4.79), we use a known technique called modified Picard iteration, where p in $k_r(p)$ is linearized by using the p at the previous time step, and implicit backward Euler discretization is employed.

First, we present a table (Table 4.1) for relative errors for the solution p comparing (1) standard MsFEM (2) MsFEM with oversampling and (3) MsFEM with limited global information at three different time instances. It is clear from this table that MsFEM with limited global information performs several times better than other local multiscale methods. This suggests that one needs to use some type of global information in constructing basis functions. We would like to note that the errors in the fluxes (defined as $k_r(p)k(x)\nabla p$) with localized methods are very large due to non-local heterogeneities. However, the multiscale finite element method with limited global information gives about 5% percent error in the fluxes. We depict the reference solution and the solution obtained using MsFEM with limited global information in Figures 4.2 - 4.7 at two different times $t = 0.12$ and $t = 0.32$. In Figures 4.2 and 4.5, we compare the solutions p at $t = 0.12$ and $t = 0.32$. This figure shows that the solution is not at steady state at $t = 0.12$. In Figures 4.3 and 4.6, the horizontal fluxes are compared, while in Figures 4.4 and 4.7, the vertical fluxes are compared. It is clear from these figures that MsFEM with limited global information provides accurate solution for such heterogeneous fields, where the localized methods fail. We would like to note that the localized methods introduce errors more than 50% in the fluxes and these fluxes are not acceptable for computational purposes.

In this section, we restricted ourselves to only a few numerical results. In particular, we have also tested MsFEM with oversampling when the oversampling region is the entire domain. This approach uses multiple global fields to represent the solution, and thus provides an accurate solution because the solution of (4.79) is a smooth functions of these fields, in general. We observed very accurate results when using MsFEM with oversampling with oversampling region being the entire domain (the errors in the solution is 1.6 %). All these results suggest that the use of limited global information is important for accurate simulation when non-local effects are strong.

Table 4.1. Relative Errors

time	conform	ovs	limited global
time=0.12	13.7%	11.5%	5.4%
time=0.2	11.1%	9.5%	3.5%
time=0.32	9.9%	8.8%	1.77%

4.6. Conclusions and Comments

In this chapter, we discuss numerical multiscale methods using global information and applications to parabolic equations. The first approach is a Galerkin multiscale finite element method using a single global field (or function) which is employed to construct multiscale finite element basis functions. Here the basis functions defined on the coarse grid are constructed from a global field which captures non-local

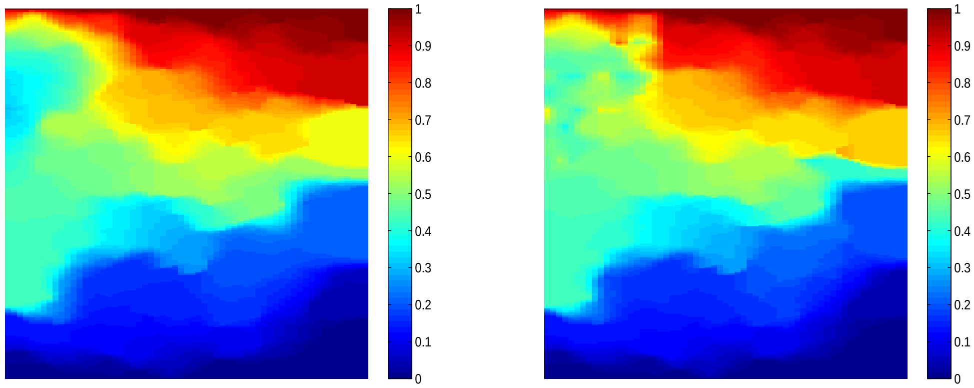

Fig. 4.2. Pressure at $t = 0.12$. Left: Reference solution. Right: The solution obtained with MsFEM using limited global information.

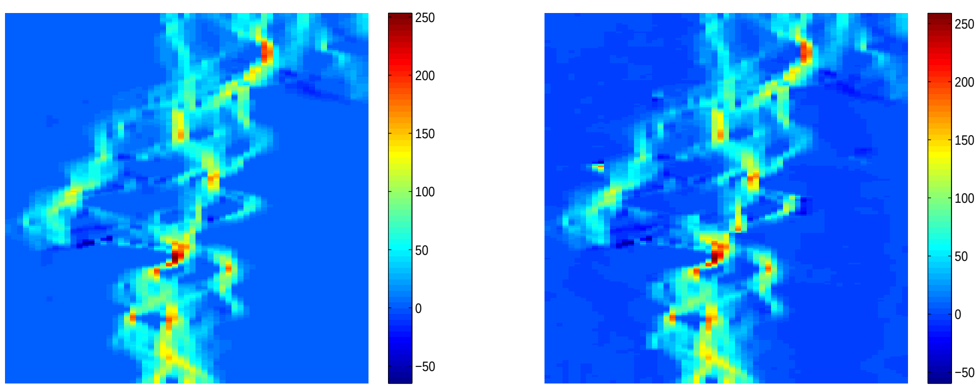

Fig. 4.3. Horizontal flux at $t = 0.12$. Left: Reference horizontal flux. Right: The horizontal flux obtained with MsFEM using limited global information.

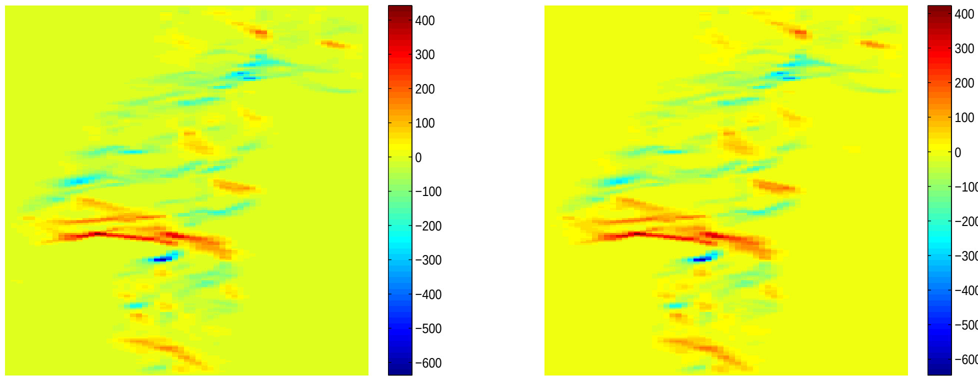

Fig. 4.4. Vertical flux at $t = 0.12$. Left: Reference vertical flux. Right: The vertical flux obtained with MsFEM using limited global information.

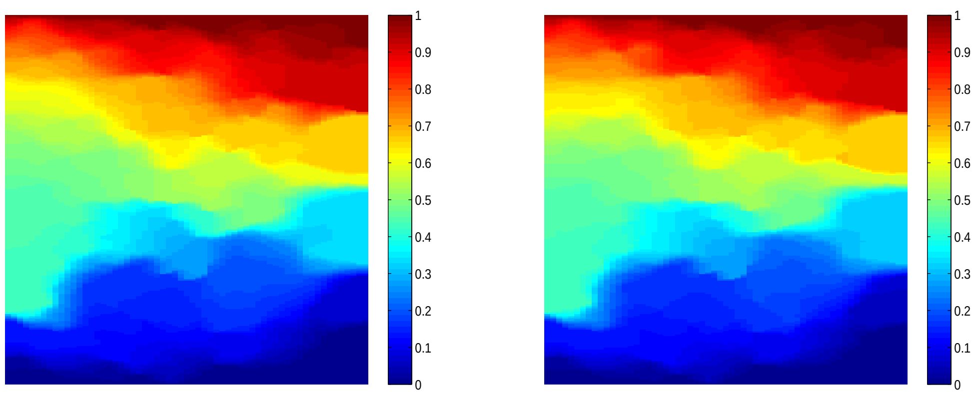

Fig. 4.5. Pressure at $t = 0.32$. Left: Reference solution. Right: The solution obtained with MsFEM using limited global information.

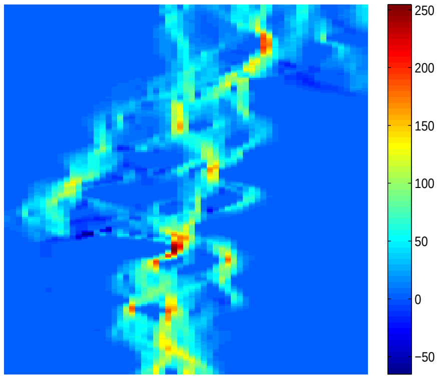

Fig. 4.6. Horizontal flux at $t = 0.32$. Left: Reference horizontal flux. Right: The horizontal flux obtained with MsFEM using limited global information.

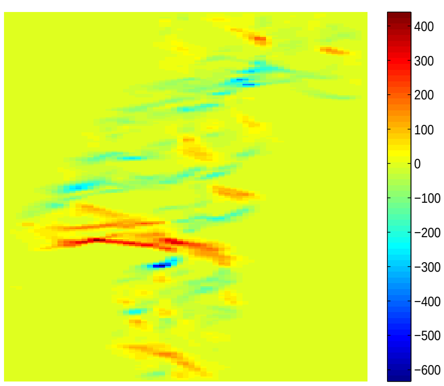

Fig. 4.7. Vertical flux at $t = 0.32$. Left: Reference vertical flux. Right: The vertical flux obtained with MsFEM using limited global information.

and small scale features of the solution. The second approach allows to incorporate multiple global fields and based on partition of unity method (PUM). Here we assume that the solution of parabolic equation smoothly depends on these global fields. The second approach is more general and capable of incorporating multiple global fields at the expense of introducing extra degrees of freedoms on each coarse element. We note that the first approach is not a special case of the second approach and it is often used in two-phase flow simulations where the single-phase flow solution is taken as a global field. Finally, we discuss a global mixed MsFEM and applications to parabolic equations. All of these multiscale approaches capture the small scale features of the solution accurately and give rise to a convergence rate independent of small scales. We provide analyses of the proposed methods and present a few numerical examples. The numerical results demonstrate that the solution can be captured more accurately on a coarse grid when some type of limited global information is used.

CHAPTER V

MULTISCALE NUMERICAL METHODS FOR WAVE EQUATIONS WITH CONTINUUM SCALES USING GLOBAL INFORMATION

In this chapter, we explore multiscale approaches for solving wave equations with heterogeneous coefficients. Our interest comes from geophysics applications and we assume that there is no scale separation with respect to spatial variables. To compute the solution of these multiscale problems on a coarse grid, we define global fields such that the solution smoothly depends on these fields. We present various multiscale finite element discretization techniques and provide analysis of these methods. A few representative numerical examples are presented using heterogeneous fields with strong non-local features. These numerical results demonstrate that the solution can be captured more accurately on the coarse grid when some type of limited global information is utilized for wave equations. The results in this chapter can been found in our paper [44].

The chapter is organized as follows.

In Section 5.1, we present acoustic wave problem setting. In Section 5.2, we present an analysis of multiscale finite element methods with limited global information. In Section 5.3, we present the analysis for a general case using partition of unity method. In Section 5.4, we present the analysis for mixed multiscale finite element method using information from multiple global fields. In Section 5.5, we present a few numerical results. Finally we make some conclusions and comments.

5.1. Preliminaries on Wave Equations

In geophysics, the scalar model acoustic problem is the following equation

$$D_{tt}p - \lambda div(\rho^{-1}\nabla p) = f, \tag{5.1}$$

where D_{tt} is the second order partial derivative regarding to t, ρ is the density of material, $\lambda > 0$ is a *Lamé* coefficient characterizing the material and p stands for the unknown pressure [41].

Without loss of generality, we will assume λ is a constant in this chapter. Our methods in the paper can be straightforwardly extended to (5.1). From now on, we consider a model wave equation

$$\begin{cases} D_{tt}p - divk(x)\nabla p &= f \quad \text{in} \quad \Omega_T \\ p(x,0) &= g_0 \quad \text{in } \Omega \\ D_t p(x,0) &= g_1 \quad \text{in } \Omega \end{cases} \tag{5.2}$$

where $\Omega_T := \Omega \times (0, T]$ and $k := k(x)$ is uniformly positive, symmetric and bounded in Ω and is a rough function with respect to x, and the problem is subject to boundary conditions. Our objective is to define multiscale methods that can capture the solution of (5.2) when $k(x)$ does not have scale separation with respect to spatial variable. In this paper, we assume that source term f, initial value g_0 and g_1 are smooth enough. This type of problems arise in many applications in geophysics, electromagnetics and seismology. Because there is no scale separation, standard upscaling and multiscale methods are not applicable. As it was shown in a number of previous findings ([53, 29]), to construct multiscale basis functions, one needs to determine global fields such that the solution smoothly depends on these fields. In this chapter, we consider an abstract framework assuming that these global fields are given. We discuss how these global fields can be determined in some special cases.

Next, we introduce some notations. For simplicity, we consider $p|_{\partial\Omega} = 0$, then the weak formulation is to find $p \in C^0(0,T;H_0^1(\Omega)) \cap C^1(0,T;L^2(\Omega))$ such that

$$\begin{cases} (D_{tt}p, v) + (k\nabla p, \nabla v) \;\; = (f, v) \\ \quad (p(0), v) = (g_0, v) \\ (D_t p(0), v) = (g_1, v), \end{cases} \tag{5.3}$$

for all $v \in H_0^1(\Omega)$, where $(.,.)$ is a standard L^2 inner product. Throughout the chapter, $D_t p(t_m) := D_t p(t)|_{t=t_m}$ at time $t_m \in [0,T]$ and the similar definition applicable to $D_{tt}p(t_m)$.

Let M be a finite dimensional subspace of $H_0^1(\Omega)$ which will be specified later. We suppose that $p_h : (0,T] \longmapsto M$ is a twice differentiable map with respect to t satisfying

$$\begin{cases} (D_{tt}p_h, v_h) + (k\nabla p_h, \nabla v_h) & = & (f, v_h) \\ (p_h(0), v_h^l) & = & (g_0, v_h^l) \\ (D_t p_h(0), v_h^l) & = & (g_1, v_h^l), \end{cases} \tag{5.4}$$

for any $v_h \in M$ and $v_h^l \in M$. Let $g_{0,h} = (g_0, v_h^l)$ and $g_{1,h} = (g_1, v_h^l)$. Let us define in this chapter

$$|||p - p_h|||_{\Omega_T}^2 = \|D_t(p - p_h)\|_{L^\infty(L^2(\Omega))}^2 + |p - p_h|_{L^\infty(H^1(\Omega))}^2.$$

After slightly modifying the proof of Lemma 1 in [28], one can obtain the following stability result.

Lemma 5.1.1. *Let p and p_h be the solutions to (5.3) and (5.4) respectively. Assume that $w : [0,T] \longmapsto M$ be any function, then*

$$\begin{aligned} |||p - p_h|||_{\Omega_T} \le C(|||p - w|||_{\Omega_T} + \|D_{tt}(p - w)\|_{L^2(L^2(\Omega))} \\ + |g_{0,h} - w(0)|_{1,\Omega} + \|g_{1,h} - D_t w(0)\|\|_{L^2(\Omega)}). \end{aligned} \tag{5.5}$$

5.2. Galerkin MsFEM with Limited Global Information

Let ϕ_i^K be defined in (3.4). We define the Galerkin finite element space by

$$V_h = span\{\phi_i^K : 1, \cdots, d; K \in \tau_h\}. \tag{5.6}$$

The weak formulation of (5.2) is to seek $u_h \in V_h \subset H_0^1$ such that

$$\begin{cases} (D_{tt}p_h, v) + (k\nabla p_h, \nabla v) &= (f, v) \\ (p_h(0) - g_0, v_h^l) &= 0 \\ (D_t p_h(0) - g_1, v_h^l) &= 0, \end{cases} \tag{5.7}$$

for all $v \in V_h$ and $v_h^l \in V_h$.

Next, following the analysis presented in [4], we assume that p smoothly depends on p_1. This can be derived for a channelized media as it is done in [29].

Assumption G. There exists a sufficiently smooth scalar valued function $G(\eta, t)$ ($G \in W^{1,\infty}(H^2) \cap H^2(H^2) \cap L^\infty(W^{3,\frac{2s}{s-2}}), s > 2$), such that

$$|||p - G(p_1, t)|||_{\Omega_T} + ||D_{tt}(u - G(p_1, t))||_{L^2(L^2(\Omega))} \leq C\delta, \tag{5.8}$$

where δ is sufficiently small.

Note that G is defined on a bounded domain because p is a bounded function and $[0, T]$ is bounded.

For an accurate numerical solution of equation (5.4) on the coarse grid, we use the global solutions (on the fine grid) of equation (4.5) to construct MsFEM basis functions as described earlier. Next, we prove that with these basis functions and *Assumption G*, MsFEM converges independent of small scales.

Theorem 5.2.1. *Under Assumption G and $p \in W^{1,s}(\Omega)$ ($s > 2$), we have*

$$|||p - p_h|||_{\Omega_T} \leq C(\delta + \alpha(h)) + Ch^{1-\frac{2}{s}}|p_1|_{1,s,\Omega}. \tag{5.9}$$

Proof. In the proof, we first use Lemma 5.1.1 and then break the estimate in this lemma into the estimates over each coarse grid block. The estimation over each coarse grid block is obtained using perturbation of G and regularities of basis functions and the global field. Following standard practice of finite element estimation, we seek finite dimensional function $w = c_i(t)\phi_i^K$, where ϕ_i^K are specified in (3.4).

From Lemma 5.1.1, we have

$$\begin{aligned}
&|||p - p_h|||_{\Omega_T} \\
&\leq C(|||p - G(p_1,t)|||_{\Omega_T} + |||G(p_1,t) - c_i(t)\phi_i^K|||_{\Omega_T} \\
&+ \|D_{tt}(p - G(p_1,t))\|_{L^2(L^2(\Omega))} + \|D_{tt}(G(p_1,t) - c_i(t)\phi_i^K)\|_{L^2(L^2(\Omega))} + \alpha(h)),
\end{aligned} \tag{5.10}$$

where we assume that

$$|g_{0,h} - c_i(0)\phi_i^K|_{1,\Omega} + \|g_{1,h} - D_t c_i(0)\phi_i^K|\|_{0,\Omega} \leq \alpha(h). \tag{5.11}$$

If g_0 and g_1 are smooth functions, we pick the basis function ϕ_i^K from S_h at $t = 0$, otherwise from V_h. Hence,

$$\begin{aligned}
|||p - p_h|||_{\Omega_T} \leq C(|||G(p_1,t) - c_i(t)\phi_i^K|||_{\Omega_T} + \|D_{tt}(G(p_1,t) - c_i(t)\phi_i^K)\|_{L^2(L^2(\Omega))}) \\
+ C\delta + C\alpha(h).
\end{aligned} \tag{5.12}$$

Next, we present an estimate for the third term. We choose $c_i(t) = G(p_1(x_i),t)$ for $t > 0$, where x_i are vertices of K. Furthermore, using Taylor expansion of G around $\overline{p_1}_K$, which is the average of p_1 over K,

$$\begin{aligned}
G(p_1(x_i),t) = G(\overline{p_1}_K,t) + \partial_\eta G(\overline{p_1}_K,t)(p_1(x_i) - \overline{p_1}_K) + \\
(p_1(x_i) - \overline{p_1}_K)^2 \int_0^1 s\partial_\eta^2 G(p_1(x_i) + s(\overline{p_1}_K - p_1(x_i)),t)ds,
\end{aligned}$$

where $\partial_\eta^2 G$ refers to the second derivative with respect to first variable $G(\eta,t)$, We have in each K

$$
\begin{aligned}
c_i(t)\phi_i^K &= G(\overline{p_1}_K, t)\sum_i \phi_i^K + \partial_\eta G(\overline{p_1}_K, t)(p_1(x_i) - \overline{p_1}_K)\phi_i^K \\
&+(p_1(x_i) - \overline{p_1}_K)^2\phi_i^K \int_0^1 s\partial_\eta^2 G(p_1(x_i) + s(\overline{p_1}_K - p_1(x_i)), t)ds \\
&= G(\overline{p_1}_K, t) + \partial_\eta G(\overline{p_1}_K, t)(p_1(x_i)\phi_i^K - \overline{p_1}_K) + \\
&(p_1(x_i) - \overline{p_1}_K)^2\phi_i^K \int_0^1 s\partial_\eta^2 G(p_1(x_i) + s(\overline{p_1}_K - p_1(x_i)), t)ds.
\end{aligned} \tag{5.13}
$$

In the last step, we have used $\sum_i \phi_i^K = 1$. Similarly, in each K,

$$
\begin{aligned}
G(p_1(x), t) &= G(\overline{p_1}_K, t) + \partial_\eta G(\overline{p_1}_K)(p_1(x) - \overline{p_1}_K) \\
&+ (p_1(x) - \overline{p_1}_K)^2 \int_0^1 s\partial_\eta^2 G(p_1(x) + s(\overline{p_1}_K - p_1(x)), t)ds.
\end{aligned} \tag{5.14}
$$

Using (5.13) and (5.14), we get

$$
\begin{aligned}
&\|D_t(G(p_1, t) - c_i(t)\phi_i^K)\|_{L^\infty(L^2(K))} \\
&\leq \|\partial_t\partial_\eta G(\overline{p_1}_K, t)(p_1(x) - p_1(x_i)\phi_i^K)\|_{L^\infty(L^2(K))} + \\
&\|(p_1(x_i) - \overline{p_1}_K)^2\phi_i^K \int_0^1 s\partial_t\partial_\eta^2 G(p_1(x_i) + s(\overline{p_1}_K - p_1(x_i)), t)ds\|_{L^\infty(L^2(K))} \\
&+ \|(p_1(x) - \overline{p_1}_K)^2 \int_0^1 s\partial_t\partial_\eta^2 G(p_1(x) + s(\overline{p_1}_K - p_1(x)), t)ds\|_{L^\infty(L^2(K))} \\
&\leq Ch^2\|f_0\|_{0,K} + Ch^{2-\frac{4}{s}}|p_1|_{1,s,K}^2 \\
&\leq Ch^2\|f_0\|_{0,K} + Ch^{2-\frac{4}{s}}|p_1|_{1,s,\Omega}|p_1|_{1,s,K}^2 \\
&\leq Ch^2\|f_0\|_{0,K} + Ch^{2-\frac{4}{s}}|p_1|_{1,s,K}.
\end{aligned} \tag{5.15}
$$

In the second step we used the fact $\|p_1(x) - p_1(x_i)\phi_i^K\|_{0,K} \leq Ch^2\|f_0\|_{0,K}$ (see [4]) , smoothness of $G(\eta, t)$(ie. $\partial_t G(\eta, t) \in L^\infty(H^2)$) and the inequality

$$|p_1(x) - p_1(y)| \leq C|x - y|^{1-\frac{2}{s}}|p_1|_{1,s,K},$$

for $x, y \in K$ when $s > 2$. A direct calculation gives

$$
\begin{aligned}
&|(p_1(x) - \overline{p_1}_K)^2 \int_0^1 s\partial_\eta^2 G(p_1(x) + s(\overline{p_1}_K - p_1(x)), t) ds|_{L^\infty(H^1(K))} \\
&\le \|(p_1(x) - \overline{p_1}_K)^2 \nabla p_1(x) \int_0^1 (1-s) s\partial_\eta^3 G(p_1(x) + s(\overline{p_1}_K - p_1(x)), t) ds\|_{L^\infty(L^2(K))} \\
&+ 2\|(p_1(x) - \overline{p_1}_K) \nabla p_1(x) \int_0^1 s\partial_\eta^2 G(p_1(x) + s(\overline{p_1}_K - p_1(x)), t) ds\|_{L^\infty(L^2(K))} \\
&\le Ch^{2-\frac{4}{s}} |p_1|_{1,s,K}^2 (|p_1|_{1,s,\Omega}^s + |G|_{L^\infty(W^{3,\frac{2s}{s-2}})}^{\frac{2s}{s-2}})^{\frac{1}{2}} \qquad (5.16) \\
&+ Ch^{1-\frac{2}{s}} |p_1|_{1,s,K} (|p_1|_{1,s,\Omega}^s + |G|_{L^\infty(W^{2,\frac{2s}{s-2}})}^{\frac{2s}{s-2}})^{\frac{1}{2}} \\
&\le Ch^{2-\frac{4}{s}} |p_1|_{1,s,K}^2 + Ch^{1-\frac{2}{s}} |p_1|_{1,s,K} \\
&\le Ch^{1-\frac{s}{2}} |p_1|_{1,s,K},
\end{aligned}
$$

where we used Young's inequality in the second step. Notice that $|p_1(x) - p_1(x_i)\phi_i^K|_{1,K} \le Ch\|f_0\|_{0,K}$ and $G \in L^\infty(H^2)$, we get

$$
\begin{aligned}
&|G(p_1, t) - c_i(t)\phi_i^K|_{L^\infty(H^1(K))} \\
&\le \|\partial_\eta G(\overline{p_1}_K, t)(p_1(x) - p_1(x_i)\phi_i^K)\|_{L^\infty(H^1(K))} + \\
&\|(p_1(x_i) - \overline{p_1}_K)^2 \phi_i^K \int_0^1 s\partial_\eta^2 G(p_1(x_i) + s(\overline{p_1}_K - p_1(x_i)), t) ds\|_{L^\infty(H^1(K))} \qquad (5.17) \\
&+ \|(p_1(x) - \overline{p_1}_K)^2 \int_0^1 s\partial_\eta^2 G(p_1(x) + s(\overline{p_1}_K - p_1(x)), t) ds\|_{L^\infty(H^1(K))} \\
&\le Ch\|f_0\|_{0,K} + Ch^{2-\frac{4}{s}} |p_1|_{1,s,K}^2 |\phi_i^K|_{1,K} + Ch^{1-\frac{s}{2}} |p_1|_{1,s,K} \\
&\le Ch\|f_0\|_{0,K} + Ch^{2-\frac{4}{s}} |p_1|_{1,s,K} + Ch^{1-\frac{s}{2}} |p_1|_{1,s,K}.
\end{aligned}
$$

In the last step, we have used the fact that $|\nabla \phi_i^K| \leq C/h$. Similarly one can estimate

$$
\begin{aligned}
&\|D_{tt}(G(p_1,t) - c_i(t)\phi_i^K)\|_{L^2(L^2(K))} \leq \|\partial_{tt}\partial_\eta G(\overline{p_1}_K, t)(p_1(x) - p_1(x_i)\phi_i^K)\|_{L^2(L^2(K))} + \\
&\|(p_1(x_i) - \overline{p_1}_K)^2 \phi_i^K \int_0^1 s\partial_{tt}\partial_\eta^2 G(p_1(x_i) + s(\overline{p_1}_K - p_1(x_i)), t)ds\|_{L^2(L^2(K))} \\
&+ \|(p_1(x) - \overline{p_1}_K)^2 \int_0^1 s\partial_{tt}\partial_\eta^2 G(p_1(x) + s(\overline{p_1}_K - p_1(x)), t)ds\|_{L^2(L^2(K))} \\
&\leq Ch^2\|f_0\|_{0,K} + Ch^{2-\frac{4}{s}}|p_1|^2_{1,s,K} \\
&\leq Ch^2\|f_0\|_{0,K} + Ch^{2-\frac{4}{s}}|p_1|_{1,s,K}.
\end{aligned} \tag{5.18}
$$

In the second step, we used the regularity $\partial_{tt}G \in L^2(H^2)$. Combing (5.15), (5.17) and (5.18), we have

$$
\begin{aligned}
&|||G(p_1,t) - c_i\phi_i^K|||_K + ||D_t(G(p_1,t) - c_i\phi_i^K)\|_{L^2(L^2(K))} \\
&\leq Ch\|f_0\|_{0,K} + Ch^{2-\frac{4}{s}}|p_1|_{1,s,K} + Ch^{1-\frac{s}{2}}|p_1|_{1,s,K}.
\end{aligned} \tag{5.19}
$$

Summing (5.19) over K and take into account (5.12), we have

$$
|||p - p_h|||_{\Omega_T} \leq C\delta + C\alpha(h) + Ch^{1-\frac{2}{s}}|p_1|_{1,s,\Omega}.
$$

Therefore, the proof is complete. □

Remark 5.2.1. *$\alpha(h)$ is the approximation errors associated with initial conditions and defined in (5.11) (see also page 219 of [36] where this term appears in standard wave equations). $\alpha(h) = 0$ if initial conditions in numerical approximation are chosen appropriately (i.e., $c_i(0) = g_{0,h}$, $D_t c_i(0) = g_{1,h}$ in (5.11)).*

Remark 5.2.2. *If $\nabla p_1 \in L^\infty(\Omega)$, the regularity of G can be relaxed to $G \in H^2(H^2) \cap L^\infty(H^3)$ and the convergence rate would be $C\delta + C\alpha(h) + Ch$.*

5.3. PUM for Wave Equations Using Multiple Global Information

5.3.1. Framework of PUM for Wave Equations

In the Section 5.2, we have studied wave equations when the solution smoothly depends on a single global field. In many applications, the solution may smoothly depend on multiple fields and we are going to discuss this case.

We define patch w_j $(j = 1, 2, ..., m)$ to be the union of elements (coarse) sharing the common vertex x_j. Let $diam(w_j) = h_j$ and $h = \max_j\{h_j\}$. We suppose that the solution p of (5.2) can be approximated by ξ_j^p on each patch w_j $(j = 1, 2, ..., m)$ and satisfies the following conditions on each patch (no summation over j),

$$\begin{aligned} \|p(t) - \xi_j^p(t)\|_{0,\omega_j}^2 &\le C\epsilon_1(t,j)^2 \\ |p(t) - \xi_j^p(t)|_{1,\omega_j}^2 &\le C\epsilon_2(t,j)^2 \\ \|D_t(p(t) - \xi_j^p(t))\|_{0,\omega_j}^2 &\le C\epsilon_3(t,j)^2 \\ \|D_{tt}(p(t) - \xi_j^p(t))\|_{0,\omega_j}^2 &\le C\epsilon_4(t,j)^2 \end{aligned} \tag{5.20}$$

Let $diam(w_j) = h_j$ and $h = \max_j\{h_j\}$. For simplicity, we still make the hypothesis,

$$|g_{0,h} - w(0)|_{1,\Omega} + \|g_{1,h} - D_t w(0)|\|_{0,\Omega} \le \alpha(h),$$

where $\alpha(h)$ is defined as before (see (5.9)).

Theorem 5.3.1. *Let p and p_h be the solutions to (5.3) and (5.4) respectively. If (5.20) holds, then*

$$|||p-p_h|||_{\Omega_T}^2 \le C\alpha(h)+C\int_0^T(\epsilon_3(t,j)^2+\epsilon_4(t,j)^2)dt+C\sup_t(\frac{\epsilon_1(t,j)^2}{h_j^2}+\epsilon_2(t,j)^2). \tag{5.21}$$

This theorem provides a convergence rate which depends on the estimates over the patches.

Remark 5.3.1. *If the patches w_j have the uniform Poincaré property (as defined in* *page 92 in [12]), one can simplify (5.21) to*

$$|||p - p_h|||^2_{\Omega_T} \le C\alpha(h) + C\int_0^T (\epsilon_3(t,j)^2 + \epsilon_4(t,j)^2)dt + \sup_t \sum_{j=1}^m \epsilon_2(t,j)^2. \qquad (5.22)$$

Remark 5.3.2. *Assume the finite space $M \in P_k$, i.e., polynomials with degree k, and u is sufficiently smooth. If w is chosen to be elliptic projection onto M of p, standard analysis and Lemma 5.1.1 or Theorem 5.3.1 imply that (see [36])*

$$|||p - p_h|||_{\Omega_T} \le C\alpha(h) + Ch^{k-1}.$$

5.3.2. Incorporating Multiple Global Information

In applications, we often deal with multiple global fields that allow to represent the solution. We assume that p can be approximated by $G(p_1, ..., p_N, t)$, where $p_i(x)$ $(i = 1, 2, ..., N)$ are pre-defined functions. Here, p, p_1, ..., p_N are scalar functions. More precisely, we assume there exists a function $G(\eta, t) \in W^{1,\infty}(H^2) \cap H^2(H^2) \cap L^\infty(W^{3,\frac{2s}{s-2}})$, $s > 2$, $\eta \in R^N$, such that

$$|||p - G(p_1, ..., p_N, t)|||_{\Omega_T} + ||D_{tt}(p - G(p_1, ..., p_N, t))||_{L^2(L^2(\Omega))} \le \delta, \qquad (5.23)$$

where $p_i \in W^{1,s}(\Omega)(s > 2)$, $i = 1, 2, ..., N$. This is an extension of the previous approach presented in Section 5.2. We would like to note that multiple global fields need to be used for stochastic differential equations ([27]). Next, we define $M = span\{p_i | i = 1, 2, ..., N\}$, $p_1 = 1$ and follow Theorem 5.3.1 to approximate p. Let $\psi_{ij} = \phi_i^0 p_j$, then $\sum_i \phi_i^0 p_j = p_j$. Next, we study this method and provide convergence analysis. Using this approach, we have the following theorem. The theorem shows that the PUM converges independent of small scales in a general setting for multiple global fields to be used.

Theorem 5.3.2. *Under the assumption (5.23) and if $s > 4$, we have*

$$|||p - p_h|||_{\Omega_T} \leq \alpha(h) + \delta + Ch^{1-\frac{4}{s}}.$$

Proof. In the proof, after formulating stability estimate (5.21), we break the estimate into the estimates over each coarse patch. The estimation over each coarse patch is obtained using perturbation of G and regularities of basis functions and the global fields. By Lemma 5.1.1, we need to estimate the right hand sides of the following inequalities

$$|||p - w|||_{\Omega_T} \leq |||p - G(p_1, ..., p_N, t)|||_{\Omega_T} + |||G(p_1, ..., p_N, t) - c_{ij}\psi_{ij}|||_{\Omega_T}, \quad (5.24)$$

and

$$\begin{aligned} ||D_{tt}(p - w)||_{L^2(L^2(\Omega))} \leq\ & ||D_{tt}(p - G(p_1, ..., p_N, t))||_{L^2(L^2(\Omega))} \\ & + ||D_{tt}(G(p_1, ..., p_N, t) - c_{ij}\psi_{ij})||_{L^2(L^2(\Omega))}, \end{aligned} \quad (5.25)$$

where $w = c_{ij}\psi_{ij}$ and c_{ij} is chosen such that the second term on the right hand of the above two inequalities is small. In each ω_i, we choose c_{ij} as

$$c_{i1} = G(\bar{p}_1^i, ..., \bar{p}_N^i, t) - \frac{\partial G}{\partial p_j}(\bar{p}_1^i, ..., \bar{p}_N^i, t)\bar{p}_j^i,$$

and

$$c_{ij} = \frac{\partial G}{\partial p_j}(\bar{p}_1^i, ..., \bar{p}_N^i, t), \quad j \geq 2,$$

where $\bar{p}_j^i$ is the average of p_j over ω_i. We note the following Taylor expansion in each ω_i,

$$G(p_1, ..., p_N, t) = G(\bar{p}_1^i, ..., \bar{p}_N^i, t) + \frac{\partial G}{\partial p_j}(\bar{p}_1^i, ..., \bar{p}_N^i, t)(p_j - \bar{p}_j^i) + R_i,$$

where R_i is the remainder, which can be written as

$$R_i = \sum_{k,j}(p_k - \bar{p}_k^i)(p_j - \bar{p}_j^i)\int_0^1 s\partial_{k,j}^2 G(p_1 + s(\bar{p}_1^i - p_1), ..., p_N + s(\bar{p}_N^i - p_N), t)ds,$$

where $\partial^2_{k,j}$ refers to second derivative with respect to p_k and p_j.

We first estimate (5.24). One can show that in each ω_i

$$
\begin{aligned}
&|||G(p_1, ..., p_N, t) - c_{ij}p_j|||_{\omega_{iT}} \\
&\leq |||G(\bar{p}^i_1, ..., \bar{p}^i_N, t) + \frac{\partial G}{\partial p_j}(\bar{p}^i_1, ..., \bar{p}^i_N, t)(p_j - \bar{p}^i_j) - c_{ij}p_j|||_{\omega_{iT}} + |||R_i|||_{\omega_{iT}},
\end{aligned}
$$

where $\omega_{iT} = \omega_i \times [0, T]$. The choice of c_{ij} implies that the first term on the right hand side is zero. We only need to estimate the second term $|||R_i|||_{\omega_{iT}}$. A straightforward computation gives rise to

$$
\begin{aligned}
\|D_t R_i\|_{L^\infty(L^2(\omega_i))} &\leq \sum_{k,j} h^{2-\frac{4}{s}} |p_k|_{1,s,\omega_i} |p_j|_{1,s,\omega_i} |D_t G|_{L^\infty(H^2)} \\
&\leq C h^{2-\frac{4}{s}} \sum_j |p_j|_{1,s,\omega_i},
\end{aligned}
\tag{5.26}
$$

$$
\begin{aligned}
\|R_i\|_{L^\infty(L^2(\omega_i))} &\leq \sum_{k,j} h^{2-\frac{4}{s}} |p_k|_{1,s,\omega_i} |p_j|_{1,s,\omega_i} |G|_{L^\infty(H^2)} \\
&\leq C h^{2-\frac{4}{s}} \sum_j |p_j|_{1,s,\omega_i}
\end{aligned}
\tag{5.27}
$$

and

$$
\begin{aligned}
|R_i|_{L^\infty(H^1(\omega_i))} &\leq \sum_{j,k,l} \|(p_j - \bar{p}^i_j)(p_k - \bar{p}^i_k)\nabla p_l \int_0^1 (1-s)s\partial^3_{k,j,l} G_s(.,t)ds\|_{L^\infty(L^2(\omega_i))} \\
&+ \sum_{j,k} \|((p_j - \bar{p}^i_j)\nabla p_j + (p_k - \bar{p}^i_k)\nabla p_k) \int_0^1 s\partial^2_{k,j} G_s(.,t)ds\|_{L^\infty(L^2(\omega_i))} \\
&\leq C \sum_{j,k,l} h^{2-\frac{4}{s}} |p_j|_{1,s,\omega_i} |p_k|_{1,s,\omega_i} (|p_l|^s_{1,s,\Omega} + |G|^{\frac{2s}{s-2}}_{L^\infty(W^{3,\frac{2s}{s-2}})})^{\frac{1}{2}} \\
&+ C \sum_j h^{1-\frac{2}{s}} |p_j|_{1,s,\omega_i} (|p_j|^s_{1,s,\Omega} + |G|^{\frac{2s}{s-2}}_{L^\infty(W^{2,\frac{2s}{s-2}})})^{\frac{1}{2}} \\
&\leq C \sum_{j,k} h^{2-\frac{4}{s}} |p_j|_{1,s,\omega_i} |p_k|_{1,s,\omega_i} + C \sum_j h^{1-\frac{2}{s}} |p_j|_{1,s,\omega_i} \\
&\leq C \sum_j h^{1-\frac{2}{s}} |p_j|_{1,s,\omega_i},
\end{aligned}
\tag{5.28}
$$

where $\partial^3_{i,j,k}$ is partial triple partial derivative with respect to p_i, p_j, p_k, $G_s(.,t) = G(p_1 + s(\bar{p}^i_1 - p_1), ..., p_N + s(\bar{p}^i_N - p_N), t)$ and Young's inequality is applied in the second step.

Note that $\sum_i \phi^0_i = 1$ and $|\nabla \phi^0_i| \leq C\frac{1}{h}$, then it follows that

$$
\begin{aligned}
&|G(p_1, ..., p_N, t) - c_{ij}\psi_{ij}|^2_{L^\infty(H^1(\Omega))} \\
&= \sup_t \int_\Omega |\nabla(G - c_{ij}\phi^0_i p_j)|^2 dx \\
&= \sup_t \int_\Omega |\nabla(\phi^0_i(G - c_{ij}p_j))|^2 dx \\
&\leq C \sup_t \int_\Omega |(G - c_{ij}p_j)\nabla\phi^0_i|^2 dx + C \sup_t \int_\Omega |\phi^0_i \nabla(G - c_{ij}p_j)|^2 dx \\
&\leq C\frac{1}{h^2}\sum_i \|R_i\|^2_{L^\infty(L^2(\omega_i))} + C\sum_i |R_i|^2_{L^\infty(H^1(\omega_i))} \\
&\leq Ch^{2-8/s}\sum_{i,j} |p_j|^2_{1,s,\omega_i} + Ch^{2-4/s}\sum_{i,j} |p_j|^2_{1,s,\omega_i} \\
&\leq Ch^{2-8/s},
\end{aligned}
\tag{5.29}
$$

and

$$
\begin{aligned}
\|D_t(G(p_1, ..., p_N, t) - c_{ij}\psi_{ij})\|^2_{L^\infty(L^2(\Omega))} &\leq C\sum_i \|D_t R_i\|^2_{L^\infty(L^2(\omega_i))} \\
&\leq Ch^{4-8/s}\sum_{i,j} |p_j|^2_{1,s,\omega_i} \\
&\leq Ch^{4-8/s}.
\end{aligned}
\tag{5.30}
$$

It remains to estimate (5.25). We only need to estimate $\|D_{tt}R_i\|_{L^2(L^2(\Omega))}$. It is easy to show

$$\begin{aligned}
\|D_{tt}R_i\|_{L^2(L^2(\omega_i))} &\le \sum_{j,k} \|(p_j - \bar{p}^i_j)(p_k - \bar{p}^i_k)\int_0^1 s\partial_{tt}\partial^2_{k,j}G_s(.,t)ds\|_{L^2(L^2(\omega_i))} \\
&\le C\sum_{j,k} h^{2-\frac{4}{s}}|p_j|_{1,s,\omega_i}|u_k|_{1,s,\omega_i}|\partial_{tt}G|_{L^2(H^2)} \\
&\le Ch^{2-\frac{4}{s}}\sum_{j,k}|p_j|_{1,s,\omega_i}|p_k|_{1,s,\Omega} \\
&\le Ch^{2-\frac{4}{s}}\sum_{j}|p_j|_{1,s,\omega_i}
\end{aligned} \tag{5.31}$$

and consequently

$$\|D_{tt}(G(p_1,...,p_N,t) - c_{ij}\psi_{ij})\|^2_{L^2(L^2(\Omega))} \le Ch^{4-8/s}.$$

Invoking (5.24) , (5.25) and Lemma 5.1.1, we complete the proof. □

Remark 5.3.3. *If $G(\eta,t) \in L^\infty(W^{2,\infty}) \cap W^{1,\infty}(H^2) \cap H^1(H^2) \cap L^\infty(W^{3,\frac{2s}{s-2}})$, $s > 2$, then the proof of Theorem 5.3.2 implies that*

$$|||p - p_h|||_{\Omega_T} \le \alpha(h) + \delta + Ch^{1-\frac{2}{s}}.$$

If patches ω_i have the uniform Poincaré property (as defined in [12]), then one can improve the convergence rate, which is the same as that in Theorem 5.2.1. The result is given in the following theorem.

Theorem 5.3.3. *Provided that patches ω_i have the uniform Poincaré property and $s > 2$, then*

$$|||p - p_h|||_{\Omega_T} \le \alpha(h) + \delta + Ch^{1-\frac{2}{s}}.$$

Proof. If patches ω_i have the uniform Poincaré property, we can reset $c_{ij}(t)p_j$ to be $c_{ij}(t)p_j + r_i(t)$ in each patch ω_i. Let $A_{int} = \{i : \bar{\omega}_i \cap \partial\Omega = 0\}$ and $A_{bd} = \{i : \bar{\omega}_i \cap \partial\Omega \neq$

0}. We choose

$$r_i(t) = \frac{1}{|\omega_i|}\int_{\omega_i}(G - c_{ij}(t)p_j)dx$$

for $i \in A_{int}$ and $r_i(t) = 0$ for $i \in A_{bd}$.

Since

$$G(\bar{p}_1^i, ..., \bar{p}_N^i, t) + \frac{\partial G}{\partial p_j}(\bar{p}_1^i, ..., \bar{p}_N^i, t)(p_j - \bar{p}_j^i) - c_{ij}p_j = 0$$

in each ω_i, it follows immediately that

$$\begin{aligned}
&\|D_t(G(p_1, ..., p_N, t) - c_{ij}p_j - r_i)\|_{L^\infty(L^2(\omega_i))}\\
&\le \|D_t(G(\bar{p}_1^i, ..., \bar{p}_N^i, t) + \frac{\partial G}{\partial p_j}(\bar{p}_1^i, ..., \bar{p}_N^i, t)(p_j - \bar{p}_j^i) - c_{ij}p_j - r_i)\|_{L^\infty(L^2(\omega_i))}\\
&+ \|D_tR_i\|_{L^\infty(L^2(\omega_i))}\\
&= \|D_tr_i\|_{L^\infty(L^2(\omega_i))} + \|D_tR_i\|_{L^\infty(L^2(\omega_))}\\
&= \|\frac{1}{|\omega_i|}\int_{\omega_i}(D_t(G - G(\bar{p}_1^i, \cdots, \bar{p}_N^i, t)) - D_t\partial_j G(\bar{p}_1^i, \cdots, \bar{p}_N^i, t)(p_j - \bar{p}_j^i))dx\|_{L^\infty(L^2(\omega_i))}\\
&+ \|D_tR_i\|_{L^\infty(L^2(\omega_i))}\\
&= \|\frac{1}{|\omega_i|}\int_{\omega_i} D_tR_i dx\|_{L^\infty(L^2(\omega_i))} + \|D_tR_i\|_{L^\infty(L^2(\omega_i))}\\
&\le \sup_t \int_{\omega_i} |D_tR_i|dx + \|D_tR_i\|_{L^\infty(L^2(\omega_i))}\\
&\le Ch\|D_tR_i\|_{L^\infty(L^2(\omega_i))} + \|D_tR_i\|_{L^\infty(L^2(\omega_i))}\\
&\le C\|D_tR_i\|_{L^\infty(L^2(\omega_i))},
\end{aligned} \tag{5.32}$$

where we used Holder inequality in the last step. Similarly, one can show

$$\|D_{tt}(G(p_1, ..., p_N, t) - c_{ij}p_j - r_i)\|_{L^2(L^2(\omega_i))} \le C\|D_{tt}R_i\|_{L^2(L^2(\omega_i))}. \tag{5.33}$$

Since patches ω_i have uniform Poincaré property by assumption in Theorem 5.3.3 , Poincaré inequality implies that there exists a constant C independent of ω_i such that

$$\begin{aligned} \|G - c_{ij}p_j - r_i\|_{L^2(\omega_i)}(t) &\leq Ch\|G - c_{ij}p_j\|_{H^1(\omega_i)}(t) \\ \|G - c_{ij}p_j - r_i\|_{H^1(\omega_i)}(t) &\leq \|G - c_{ij}p_j\|_{H^1(\omega_i)}(t). \end{aligned} \tag{5.34}$$

From (5.29) and (5.34) , it follows that

$$\begin{aligned} &|G(p_1, ..., p_N, t) - c_{ij}(t)\psi_{ij} - \phi_i^0 r_i|^2_{L^\infty(H^1(\Omega))} \\ &= \sup_t \int_\Omega |\nabla(\phi_i^0((G - c_{ij}p_j - r_i))|^2 dx \\ &\leq C \sup_t \int_\Omega |(G - c_{ij}p_j - r_i)\nabla\phi_i^0|^2 dx + C \sup_t \int_\Omega |\phi_i^0 \nabla(G - c_{ij}p_j - r_i)|^2 dx \\ &\leq C \frac{1}{h^2} \sup_t \sum_i h^2 \|G - c_{ij}p_j\|^2_{H^1(\omega_i)} + C \sup_t \sum_i |G - c_{ij}p_j|^2_{H^1(\omega_i)} \\ &\leq C(\sum_i (|R_i|_{L^\infty(H^1(\omega_i))})^2) \\ &\leq C(\sum_i (\sum_j h^{1-\frac{2}{s}} |p_j|_{1,\omega_i})^2) \\ &\leq Ch^{2-\frac{4}{s}}. \end{aligned} \tag{5.35}$$

By (5.32) and (5.30), we obtain that

$$\begin{aligned} \|D_t(G(p_1, ..., p_N, t) - c_{ij}\psi_{ij} - \phi_i^0 r_i)\|_{L^\infty(L^2(\Omega))} &\leq C \sum_i \|D_t R_i\|_{L^\infty L^2(\omega_i)} \\ &\leq Ch^{2-\frac{4}{s}}. \end{aligned} \tag{5.36}$$

By (5.33) and (5.31), it follows that

$$\begin{aligned} \|D_{tt}(G(p_1, ..., p_N, t) - c_{ij}\psi_{ij} - \phi_i^0 r_i)\|_{L^2(L^2(\Omega))} &\leq C \sum_i \|D_t R_i\|_{L^2(L^2(\omega_i))} \\ &\leq Ch^{2-\frac{4}{s}}. \end{aligned} \tag{5.37}$$

Invoking (5.35), (5.36), (5.37) and Lemma 5.1.1, the proof is complete. $\square$

Remark 5.3.4. *Let* p_i $(i = 1, 2, \cdots, dim(\Omega))$ *be the solution of the following equation*

$$\begin{aligned} div k(x)\nabla p_i &= 0 \quad in \quad \Omega \\ p_i(x) &= x_i \quad on \quad \partial\Omega. \end{aligned} \tag{5.38}$$

Define a map $F : x \longrightarrow (p_1(x), \cdots, p_n(x))$*, authors in [51] showed* $p(F^{-1}(p_1, \cdots, p_n), t) \in L^\infty(H^2)$ *(or* $W^{1,\infty}(H^2)$ *if* f *and initial conditions are sufficiently smooth) and solved the acoustic wave equations in* $(p_1, \cdots, p_n)$ *harmonic coordinate system. Let a finite dimensional space be defined by*

$$\hat{M} = \{\varphi(F(x)) : \varphi \in X_h\},$$

where X_h *is some standard finite element space, e.g., piecewise linear finite element space or weighted extended B-splines space [37]. If* $p_h \in \hat{M}$*, then it is shown in [51] that*

$$|||p - p_h|||_{\Omega_T} \le Ch.$$

5.3.3. Time Discretization of PUM for Wave Equations

In the previous subsection, we have considered the semi-discretization of the wave equation (5.4) using Galerkin MsFEM and PUM. In practice, one also needs to discretize the temporal variables. In this section, we present time discretization of multiscale wave equation when multiple global fields are used to construct basis functions. We introduce the following notations. If $p_h : \{t_m\}_0^J \longrightarrow X$, where J is a positive integer. Let $\Delta t = \frac{T}{J}$ and p^n be the value of p at $t = n\Delta t$. We will use the following notations.

$$\begin{aligned}
p^{n+\frac{1}{2}} &= \frac{p^{n+1}+p^n}{2} \\
p^{n,\frac{1}{4}} &= \frac{1}{4}p^{n+1}+\frac{1}{2}p^n+\frac{1}{4}p^{n-1} \\
D_t p^{\frac{1}{2}} &= \frac{p^1-p^0}{\Delta t} \\
D_t p^n &= \frac{p^{n+1}-p^{n-1}}{2\Delta t} \\
D_{tt} p^n &= \frac{p^{n+1}-2p^n+p^{n-1}}{\Delta t^2}.
\end{aligned}$$

We use the discrete time Galerkin method to (5.4)

$$(D_{tt}p_h^m, v_h) + (k\nabla p_h^{m,\frac{1}{4}}, \nabla v_h) = (f^{m,\frac{1}{4}}, v_h), \quad \forall v_h \in M, \tag{5.39}$$

where we assume that f is sufficiently smooth with respect to temporal variable. The scheme in (5.39) is unconditionally stable [28]. In this subsection, we always assume that p_h is the solution of (5.39), unless otherwise is stated. Suppose w is an arbitrary map from $[0,t]$ into M, and $\xi = p_h - w$, $\eta = p - w$ and consistency error

$$R^m = D_{tt}p^m - (\frac{1}{4}D_t p(t_{m+1}) + \frac{1}{2}D_t p(t_m) + \frac{1}{4}D_t p(t_{m-1})|),$$

then we obtain that

$$(D_{tt}\xi^m, v_h) + (k\nabla \xi^{m,\frac{1}{4}}, \nabla v_h) = (D_{tt}\eta^m + k\nabla\eta^{m,\frac{1}{4}} + R^m, \nabla v_h), \quad v_h \in M.$$

Let us define

$$\|p\|_{L^\infty_\Delta(X)} = \max_n \|p^{n+\frac{1}{2}}\|_X$$

for some normed space X and

$$|||p-p_h|||_{\Omega_T,\Delta} = \|D_t(p-p_h)\|_{L^\infty_\Delta(L^2(\Omega))} + |p-p_h|_{L^\infty_\Delta(H^1(\Omega))}.$$

Take $v_h = D_t\xi^m$ and apply discrete Grownwall's lemma, the proof of Lemma 6 in [28] implies that

$$\begin{aligned}\|D_t\xi\|_{L^\infty_\Delta(L^2(\Omega))} + \|\xi\|_{L^\infty_\Delta(H^1(\Omega))} \le C(&\|D_t\xi^{\frac{1}{2}}\|_{0,\Omega} + \|\xi^{\frac{1}{2}}\|_{1,\Omega} \\ &+\|D_{tt}\eta\|_{0,\Omega} + |\eta|_{L^\infty(H^1(\Omega))} + \|D_t^4 p\|_{L^2(L^2(\Omega))}\Delta t^2).\end{aligned}$$

Triangle inequality gives

$$\begin{aligned}|||p - p_h|||_{\Omega_T,\Delta} \le C(&\|D_t\xi^{\frac{1}{2}}\|_{0,\Omega} + \|\xi^{\frac{1}{2}}\|_{1,\Omega} \\ &+\|D_{tt}(p - w)\|_{0,\Omega} + |||p - w|||_{\Omega_T,\Delta} + \|D_t^4 p\|_{L^2(L^2(\Omega))}\Delta t^2).\end{aligned}$$

Consequently, we have the following theorem. This theorem gives a spatio-temporal convergence rate through the the approximation estimates on the patches.

Theorem 5.3.4. *Let p and p_h^m be the solutions to (5.3) and (5.39) respectively. Suppose $\|D_t\xi^{\frac{1}{2}}\|_{0,\Omega} + \|\xi^{\frac{1}{2}}\|_{1,\Omega} \le \alpha(h)$ and p is sufficiently smooth for t, then*

$$\begin{aligned}|||p - p_h|||^2_{\Omega_T,\Delta} \le C(\alpha(h) + \Delta t^2) &+ C\int_0^T (\epsilon_3(t, j)^2 + \epsilon_4(t, j)^2)dt \\ &+ C\sup_t(\frac{\epsilon_1(t, j)^2}{h_j^2} + \epsilon_2(t, j)^2).\end{aligned} \tag{5.40}$$

Applying Theorem 5.3.4 and repeat the same procedure as that in the proof of Theorem 5.3.2, and we have the following theorem immediately.

Theorem 5.3.5. *Under assumptions in Theorem 5.3.2 and Theorem 5.3.4, we have*

$$|||p - p_h|||_{\Omega_T,\Delta} \le \alpha(h) + \delta + C(h^{1-\frac{4}{s}} + \Delta t^2).$$

Following the similar analysis presented in Theorem 5.3.3, one can obtain the following theorem.

Theorem 5.3.6. *Provided that patches ω_i have the uniform Poincaré property and $s > 2$, then*

$$|||p - p_h|||_{\Omega_T,\Delta} \le \alpha(h) + \delta + C(h^{1-\frac{2}{s}} + \Delta t^2).$$

If we define $\zeta = p$ and $\rho = D_t p$, then one can rewrite (5.3) as a system

$$\begin{aligned} (D_t\rho, v) + (a\nabla\zeta, v) &= (f, v), \quad \forall v \in H_0^1 \\ (D_t\zeta, v) - (\rho, v) &= 0, \quad \forall v \in H_0^1, \end{aligned}$$

and its full discretization form is

$$\begin{cases} (\frac{\rho_h^{n+1}-\rho_h^n}{\Delta t}, v_h) + (a\beta\zeta_h^{n+1} + (1-\beta)\zeta_h^n, v_h) &= (f, v_h) \\ (\frac{\zeta_h^{n+1}-\zeta_h^n}{\Delta t}, v_h) - (\gamma\rho_h^{n+1} + (1-\gamma)\rho_h^n, v_h) &= 0 \\ (\zeta_h^0 - g_0, v_h) &= 0 \\ (\rho_h^0 - g_1, v_h) &= 0 \end{cases} \tag{5.41}$$

for $\forall v_h \in M$. It is known that the scheme in (5.41) is unconditionally stable when $\beta \geq \frac{1}{2}$ and $\gamma \geq \frac{1}{2}$ [55]. In the numerical experiment section, we will use this scheme to discretize the time.

5.4. Mixed MsFEM for Wave Equations Using Multiple Global Information

In this section, we apply a mixed MsFEM using multiple global fields we proposed in [5]. This approach is a mass conservative approach.

Let $u = k\nabla p$ and assume zero Dirichlet boundary conditions for (5.2). The weak mixed formulation is to find $\{p, u\} : (0, T] \longrightarrow L^2(\Omega) \times H(div, \Omega)$ such that

$$\begin{cases} (D_{tt}p, w) - (divu, w) &= (f, w) \quad \forall w \in L^2(\Omega) \\ (k^{-1}u, \chi) + (p, div\chi) &= 0 \quad \forall \chi \in H(div, \Omega) \\ (p(0), w) &= (g_0, w) \quad \forall w \in L^2(\Omega) \\ ((D_t p)(0), w) &= (g_1, w) \quad \forall w \in L^2(\Omega) \\ (k^{-1}u(0), \chi) &= (\nabla g_0, \chi) \quad \forall \chi \in H(div, \Omega). \end{cases} \tag{5.42}$$

Assumption A1w. There exist functions $u_1, \cdots, u_N$ and $A_1(t, x), \cdots, A_N(t, x)$

such that

$$u(t,x) = A_i(t,x)u_i(x),$$

where $A_i(t,x)$'s are smooth functions (specify their smoothness later) and $u_i = k(x)\nabla p_i$ ($i = 1, ..., N$) solves a elliptic equation $divk(x)\nabla p_i = 0$ with appropriate boundary conditions.

To numerically approximate the mixed problem (5.42), we construct the basis function ϕ_{ij}^K defined in (3.120) in Section 3.3.3 for the velocity, Note that for each edge e, we have N basis functions and we assume that u_1,..., u_N are linearly independent in order to guarantee that the basis functions are linearly independent. To avoid the possibility that $\int_{e_l} u_i \cdot nds$ is zero or unbounded, we make the following assumption for convenience analysis.

Assumption A2w. There exist positive constants C such that

$$\int_{e_l} |u_i \cdot n| ds \le Ch^{\beta_1} \quad \text{and} \quad \| \frac{u_i \cdot n}{\int_{e_l} u_i \cdot nds} \|_{L^r(e_l)} \le Ch^{-\beta_2 + \frac{1}{r} - 1}$$

uniformly for all edges e_l, where $\beta_1 \le 1$, $\beta_2 \ge 0$, and $r \ge 1$.

We define $\psi_{ij}^K = k(x)\nabla\phi_{i,j}^K$ and

$$\Sigma_h = \bigoplus_K \{\psi_{ij}^K\} \subset H(div, \Omega),$$

where ϕ_{ij}^K is defined in (3.120). Let $Q_h = \bigoplus_K \mathrm{P}_0(K) \subset L^2(\Omega)$, i.e., piecewise constants, be the basis functions approximating u. Let P_h be $L^2(\Omega)$ orthogonal projection onto Q_h. Denote by R_j^K be the lowest Raviart-Thomas basis function [19] associated edge e_j of K. For $t = 0$, we define

$$\Pi_h|_K u(0) = (\int_{e_j} k(x)\nabla g_0 \cdot ndx)R_j^K$$

in each element K. For $t > 0$, we define

$$\Pi_h|_K u(t) = (\int_{e_j} A_i(t,x) u_i \cdot n dx)\psi_{ij}^K$$

in each element K. If $A_i(t,x)$ and u_i satisfy proper smoothness, i.e., $A_i(t,.) \in W^{1,\xi}(\Omega)$ and $u_i \in L^\eta(\Omega)$ $(\frac{1}{2} = \frac{1}{\xi} + \frac{1}{\eta})$, then Π_h is well defined (see [19, 5]).

The numerical weak mixed formulation is to find $\{p_h, u_h\} : (0,T] \longrightarrow Q_h \times \Sigma_h$ such that

$$\begin{cases} (D_{tt}p_h, w) - (divu_h, w) &= (f, w) \quad \forall w \in Q_h \\ (k^{-1}u_h, \chi) + (p_h, div\chi) &= 0 \quad \forall \chi \in \Sigma_h \\ (p_h(0), w) &= (g_0, w) \quad \forall w \in Q_h \\ ((D_t p_h)(0), w) &= (g_1, w) \quad \forall w \in Q_h \\ (u_h(0), \chi) &= (\Pi_h u(0), \chi) \quad \forall \chi \in RT_h^0, \end{cases} \tag{5.43}$$

where RT_h^0 is the lowest Raviart-Thomas space.

We define $\|u\|_{L_k^2(\Omega)}$ and $\|u\|_{L^2(0,T;L_k^2(\Omega))}$ in the same way as in Section 4.4. Let us recall some results from [5].

$$u_i|_K = \beta_{ij}^K \psi_{ij}^K, \tag{5.44}$$

where $\beta_{ij}^K = \int_{e_j} \delta_{ij} u_i \cdot n dx$ (summation convention is not applied here) (see Lemma 3.1 in [5]). For the property of Π_h, Lemma 3.2 in [5] claims

$$(\nabla \cdot (u - \Pi_h u), w) = 0 \quad w \in Q_h. \tag{5.45}$$

Based on *Assumption A1w* and the construction of the velocity basis given in (3.120), the convergence rate by the mixed method is given in the following theorem.

Theorem 5.4.1. *Let* $\{p, u\}$ *and* $\{p_h, u_h\}$ *be respectively solution of (5.42) and (5.43). If Assumption A1w and Assumption A2w hold and* $A_i(t,x) \in L^2(C^\alpha(\Omega))$ *for* $i =$

$1, \cdots, N,$ *then for* $\alpha + \beta_1 - \beta_2 - 1 > 0$,

$$\|p - p_h\|_{L^\infty(L^2(\Omega))} + \sup_t \| \int_0^t (u(s) - u_h(s))ds\|_{L^2_k(\Omega)} \le Ch^{\min(1,\alpha+\beta_1-\beta_2-1)}.$$

Proof. The first two equations in (5.42) and (5.43) imply that

$$\begin{aligned} (D_{tt}(p - p_h), w) - (\nabla \cdot (u - u_h), w) = 0 \quad \forall w \in Q_h \\ (k^{-1}(u - u_h), \chi) + (p - p_h, \nabla \cdot \chi) = 0 \quad \forall \chi \in \Sigma_h. \end{aligned} \tag{5.46}$$

Since $p - p_h = p - P_h p + P_h p - p_h$ and $u - u_h = u - \Pi_h u + \Pi_h u - u_h$, a straightforward calculation gives for $\forall w \in Q_h$ and $\forall \chi \in \Sigma_h$

$$\begin{aligned} (D_{tt}(P_h p - p_h), w) - (\nabla \cdot (\Pi_h u - u_h), w) = (D_{tt}(P_h p - p), w) - (\nabla \cdot (\Pi_h u - u), w) \\ (k^{-1}(\Pi_h u - u_h), \chi) + (P_h p - p_h, \nabla \cdot \chi) = (k^{-1}(\Pi_h u - u), \chi) + (P_h p - p, \nabla \cdot \chi). \end{aligned} \tag{5.47}$$

Since P_h is a L^2 projection and (5.45) holds, then (5.47) reduces to

$$\begin{aligned} (D_{tt}(P_h p - p_h), w) - (\nabla \cdot (\Pi_h u - u_h), w) = (D_{tt}(P_h p - p), w) \quad \forall w \in Q_h \\ (k^{-1}(\Pi_h u - u_h), \chi) + (P_h p - p_h, \nabla \cdot \chi) = (k^{-1}(\Pi_h u - u), \chi) \quad \forall \chi \in \Sigma_h. \end{aligned} \tag{5.48}$$

If one integrates the first equation with respect to s from 0 to t and then set $w = P_h p - p_h$, one can get

$$\begin{aligned} (D_t(P_h p - p_h), P_h p - p_h) - (\textstyle\int_0^t \nabla \cdot (\Pi_h u(s) - u_h(s))ds, P_h p - p_h) \\ = (D_t(P_h p - p), P_h p - p_h) - (D_t(P_h p - p)(0), P_h p - p_h), \end{aligned}$$

where we have used the fact $(D_t(P_h p - p_h)(0), w) = 0$ by (5.43). Because P_h commutes with D_t, it follows

$$(D_t(P_h p - p_h), P_h p - p_h) - (\int_0^t \nabla \cdot (\Pi_h u(s) - u_h(s))ds, P_h p - p_h) = 0. \tag{5.49}$$

Pick $\chi = \int_0^t (\Pi_h u(s) - u_h(s))ds$ in the second equation in (5.48), then

$$\begin{aligned}&(k^{-1}(\Pi_h u - u_h), \int_0^t (\Pi_h u(s) - u_h(s))ds) + (P_h p - p_h, \int_0^t \nabla\cdot(\Pi_h u(s) - u_h(s))ds)\\ &= (k^{-1}(\Pi_h u - u), \int_0^t (\Pi_h u(s) - u_h(s))ds).\end{aligned} \tag{5.50}$$

Summing (5.49) and (5.50), we have

$$\begin{aligned}&(D_t(P_h p - p_h), P_h p - p_h) + (k^{-1}(\Pi_h u - u_h), \int_0^t (\Pi_h u(s) - u_h(s))ds)\\ &= (k^{-1}(\Pi_h u - u), \int_0^t (\Pi_h u(s) - u_h(s))ds).\end{aligned} \tag{5.51}$$

Applying integration by parts and Schwarz inequality to (5.51), we obtain

$$\begin{aligned}&D_t\|P_h p - p_h\|^2_{L^2(\Omega)} + D_t\|\textstyle\int_0^t (\Pi_h u(s) - u_h(s))ds\|^2_{L^2_k(\Omega)}\\ &\le \|\Pi_h u - u\|^2_{L^2_k(\Omega)} + \|\textstyle\int_0^t (\Pi_h u(s) - u_h(s))ds\|^2_{L^2_k(\Omega)}.\end{aligned}$$

Integrating the above inequality with respect to time from 0 to t and noting that $\|P_h p - p_h\|^2_{L^2(\Omega)}(0) = 0$ from (5.43), then we apply Gronwall's inequality to get

$$\|P_h p - p_h\|^2_{L^2(\Omega)}(t) + \|\int_0^t (\Pi_h u(s) - u_h(s))ds\|^2_{L^2_k(\Omega)} \le C\|\Pi_h u - u\|^2_{L^2(0,t;L^2_k(\Omega))}. \tag{5.52}$$

Consequently,

$$\|P_h p - p_h\|^2_{L^\infty(L^2(\Omega))} + \sup_t \|\int_0^t (\Pi_h u(s) - u_h(s))ds\|^2_{L^2_k(\Omega)} \le C\|\Pi_h u - u\|^2_{L^2(L^2_k(\Omega))}. \tag{5.53}$$

Invoking $p - p_h = p - P_h p + P_h p - p_h$ and $\int_0^t (u - u_h)ds = \int_0^t (u - \Pi_h u)ds + \int_0^t (\Pi_h u - u_h)ds$, triangle inequality infers that

$$
\begin{aligned}
&\|p-p_h\|^2_{L^\infty(L^2(\Omega))}+\sup_t\|\int_0^t(u(s)-u_h(s))ds\|^2_{L^2_k(\Omega)}\\
&\le C(\|P_hp-p\|^2_{L^\infty(L^2(\Omega))}+\sup_t\|\int_0^t(\Pi_hu(s)-u(s))ds\|^2_{L^2_k(\Omega)}+\|\Pi_hu-u\|^2_{L^2(L^2_k(\Omega))})\\
&\le C(\|P_hp-p\|^2_{L^\infty(L^2(\Omega))}+\sup_t t\|\Pi_hu-u\|^2_{L^2(0,t;L^2_k(\Omega))}+\|\Pi_hu-u\|^2_{L^2(L^2_k(\Omega))})\\
&\le C(\|P_hp-p\|^2_{L^\infty(L^2(\Omega))}+\|\Pi_hu-u\|^2_{L^2(L^2_k(\Omega))}),
\end{aligned}
\tag{5.54}
$$

where we have used Jensen's inequality in the second step.

If the source term $f\in L^2(L^2(\Omega))$, the initial conditions $g_0\in H^1(\Omega)$ and $g_1\in L^2(\Omega)$, then $p\in L^\infty(H^1(\Omega))$ (see [51, 35]). Thanks to the fact that P_h is the $L^2(\Omega)$ projection onto Q_h,

$$
\|p-P_hp\|_{L^\infty(L^2(\Omega))}\le Ch|p|_{L^\infty(H^1(\Omega))}, \tag{5.55}
$$

this estimates the first term of right hand side in (5.53). Next we estimate the term $\|u-\Pi_hu\|^2_{L^2(L^2_k(\Omega))}$. Define

$$
A^K_{ij}(t)=\int_{e_j}A_i(t,s)u_i\cdot nds
$$

in each element K. Let $\bar{A}^j_i$ be the average $A_i(x)$ along e_j, then

$$
\begin{aligned}
|A^K_{ij}-\bar{A}^j_i\beta^K_{ij}|&=|\int_{e_j}A_iu_i\cdot nds-\bar{A}^j_i\int_{e_j}u_i\cdot nds|\\
&\le Ch^{\alpha+\beta_1}\|A_i(t)\|_{C^\alpha(\Omega)},
\end{aligned}
\tag{5.56}
$$

where we have used the *Assumption A2w*.

Invoking *Assumption A1w*, (5.44) and $\|\psi^K_{ij}\|_{0,K}\le C(1+h^{-\beta_2})$ [5], we have in each element K (summation convention is used in the following)

$$
\begin{aligned}
&\|u-\Pi_h u\|^2_{L^2(0,T;L^2_k(K))} = \\
&\int_0^T\int_K (A_i(t,x)\beta^K_{ij} - A^K_{ij}(t))\psi^K_{ij}\cdot k^{-1}(A_i(t,x)\beta^K_{ij} - A^K_{ij}(t))\psi^K_{ij}dxdt \\
&\leq C\int_0^T\int_K ((A_i(t,x)\beta^K_{ij} - A^K_{ij}(t))\psi^K_{ij})^2 dxdt \\
&= C\|(A_i(t,x)\beta^K_{ij} - A^K_{ij}(t))\psi^K_{ij}\|^2_{L^2(0,T;L^2(K))} \\
&\leq C\|(A_i(t,x) - \bar{A}^j_i(t))\beta^K_{ij}\psi^K_{ij}\|^2_{L^2(0,T;L^2(K))} \\
&+ C\|(\bar{A}^j_i(t)\beta^K_{ij} - A^K_{ij}(t))\psi^K_{ij}\|^2_{L^2(0,T;L^2(K))} \\
&\leq Ch^{2(\alpha+\beta_1)}(\sum_i \|A_i\|^2_{L^2(0,T,C^\alpha(K))})\sum_{ij}\|\psi^K_{ij}\|^2_{0,K} \\
&\leq Ch^{2(\alpha+\beta_1-\beta_2)}(\sum_i \|A_i\|^2_{L^2(0,T,C^\alpha(K))}),
\end{aligned}
\tag{5.57}
$$

where we have used facts that $A_i \in L^2(0,T;C^\alpha(\Omega))$ and (5.56). After making summation over all K for (5.57), we have

$$
\|u-\Pi_h u\|_{L^2(0,T;L^2_k(\Omega))} \leq Ch^{\alpha+\beta_1-\beta_2-1}. \tag{5.58}
$$

Taking into account (5.55) , (5.58) and (5.53), the proof is complete.

□

Remark 5.4.1. *From (5.57), it is sufficient to assume that $A_i(t,.)$ is x-pointwise Holder continuous for all t and $A_i(t,x)$ is bounded globally. If global fields p_i ($i = 1,\cdots,N$, where $N = dim(\Omega)$) are defined in (5.38) and set $p = p(t,p_1,\cdots,p_N)$, then*

$$
k\nabla p = \frac{\partial p}{\partial p_i}k\nabla p_i := A_i(t,x)u_i,
$$

where $A_i(t,x) = \frac{\partial p}{\partial p_i}$ and $u_i = k\nabla p_i$. Provided that $f \in L^\infty(L^p(\Omega)) \cap H^1(L^p(\Omega))$, $g_1 \in W^{1,p}(\Omega)$ and $D_{tt}p(0) \in L^p(\Omega)$, then the proof Theorem 1.1 in [51] implies that $A_i(t,x) = \frac{\partial p}{\partial p_i} \in L^\infty(W^{1,p}(\Omega))$. Consequently $A_i(t,x) \in L^2(C^{1-\frac{N}{p}}(\Omega))$ if $p > N$ by *using Sobolev embedding theorem.*

Remark 5.4.2. *In the case of homogenization problems with ϵ-period, $A_i(t,x) = \frac{\partial p^*}{\partial x_i}$ and $u_i = k_\epsilon(x)\nabla p_i$. Here p^* is the solution of the homogenized wave equation and p_i are computed in each ϵ-period [5].*

If the functions $A_i(t,x)$ in *Assumption A1w* have higher regularity regarding to time t, we can obtain an convergence rate in energy norm, the definition of which is similar to $|||p - p_h|||_{\Omega_T}$ defined in section 5.1. The following is the result.

Theorem 5.4.2. *Let $\{p, u\}$ and $\{p_h, u_h\}$ be respectively solution of (5.42) and (5.43). If Assumption A1w holds and $A_i(t,x) \in L^\infty(C^\alpha(\Omega)) \cap H^1(C^\alpha(\Omega))$ for $i = 1, \cdots, N$, then for $\alpha + \beta_1 - \beta_2 - 1 > 0$,*

$$\|p - p_h\|_{L^\infty(L^2(\Omega))} + \|u - u_h\|_{L^\infty(L^2_k(\Omega))} \leq Ch^{\min(1,\alpha+\beta_1-\beta_2-1)}.$$

Proof. We take $w = D_t(P_h p - p_h)$ and $\chi = \Pi_h u - u_h$ in (5.48). Since D_t commutes with P_h and (5.45) holds, it follows

$$\begin{aligned} &(D_{tt}(P_h p - p_h), D_t(P_h p - p_h)) - (\nabla \cdot (\Pi_h u - u_h), D_t(P_h p - p_h)) = 0 \\ &(k^{-1}(D_t(\Pi_h u - u_h), \Pi_h u - u_h) + (D_t(P_h p - p_h), \nabla \cdot (\Pi_h u - u_h)) \\ &= (k^{-1}(D_t(\Pi_h u - u)), \Pi_h u - u_h), \end{aligned} \tag{5.59}$$

where we differentiated the second equation in (5.48) to get the second equation of (5.59). Make summation of the two equation in (5.59), we have

$$\begin{aligned} &(D_{tt}(P_h p - p_h), D_t(P_h p - p_h)) + (k^{-1}(D_t(\Pi_h u - u_h), \Pi_h u - u_h) \\ &= (k^{-1}(D_t(\Pi_h u - u)), \Pi_h u - u_h). \end{aligned} \tag{5.60}$$

After applying integration by parts and Schwarz inequality to (5.60), it follows

$$D_t\|D_t(P_h p - p_h)\|^2_{L^2(\Omega)} + D_t\|\Pi_h u - u_h\|^2_{L^2_k(\Omega)} \leq \|D_t(\Pi_h u - u)\|^2_{L^2_k(\Omega)} + \|\Pi_h u - u_h\|^2_{L^2_k(\Omega)}.$$

By integrating with respect to t in the above, Gronwall's inequality gives rise to

$$\begin{aligned}
&\|D_t(P_hp - p_h)\|^2_{L^\infty(L^2(\Omega))} + \|\Pi_h u - u_h\|^2_{L^\infty(L^2_k(\Omega))} \\
&\le C(\|D_t(P_hp - p_h)\|^2_{L^2(\Omega)}(0) + \|\Pi_h u - u_h\|^2_{L^2_k(\Omega)}(0)) \\
&\le C\|D_t(\Pi_h u - u)\|^2_{L^2(L^2_k(\Omega))}.
\end{aligned} \tag{5.61}$$

From the fourth and fifth equation of (5.43), the first term in right hand of (5.61) is vanished and (5.61) becomes

$$\begin{aligned}
&\|D_t(P_hp - p_h)\|^2_{L^\infty(L^2(\Omega))} + \|\Pi_h u - u_h\|^2_{L^\infty(L^2_k(\Omega))} \\
&\le C\|D_t(\Pi_h u - u)\|^2_{L^2(L^2_k(\Omega))}.
\end{aligned} \tag{5.62}$$

Owing to triangle inequality and (5.62), it follows immediately that

$$\begin{aligned}
&\|D_t(p - p_h)\|_{L^\infty(L^2(\Omega))} + \|\Pi_h u - u_h\|_{L^\infty(L^2_k(\Omega))} \\
&\le C(\|D_t(P_hp - p)\|_{L^\infty(L^2(\Omega))} + \|D_t(\Pi_h u - u)\|_{L^2(L^2_k(\Omega))} + \|\Pi_h u - u\|_{L^\infty(L^2_k(\Omega))}).
\end{aligned} \tag{5.63}$$

If source term $f \in H^1(L^2(\Omega))$, initial value $g_1 \in H^1(\Omega)$ and $D_{tt}p(0) \in L^2(\Omega)$, then $D_tp \in L^\infty(H^1(\Omega))$ (see [51, 35]). Since P_h is the $L^2(\Omega)$ projection onto Q_h,

$$\|D_t(p - P_hp)\|_{L^\infty(L^2(\Omega))} \le Ch|D_tp|_{L^\infty(H^1(\Omega))}, \tag{5.64}$$

If $A_i(t,x) \in L^\infty(C^\alpha(\Omega)) \cap H^1(C^\alpha(\Omega))$, one can utilizing the similar process in (5.57) to get

$$\begin{aligned}
\|\Pi_h u - u\|_{L^\infty(L^2_k(\Omega))} &\le Ch^{\alpha+\beta_1-\beta_2-1} \\
\|D_t(\Pi_h u - u)\|_{L^2(L^2_k(\Omega))} &\le Ch^{\alpha+\beta_1-\beta_2-1}.
\end{aligned} \tag{5.65}$$

Combining (5.63), (5.64) and (5.65), the proof is complete. $\square$

Remark 5.4.3. *If global fields p_i $(i = 1, \cdots, N, \quad N = dim(\Omega))$ are defined in (5.38)* *and set* $p = p(t, p_1, \cdots, p_N)$, *then*

$$k\nabla p = \frac{\partial p}{\partial p_i} k\nabla p_i := A_i(t,x)u_i,$$

where $A_i(t,x) = \frac{\partial p}{\partial p_i}$ *and* $u_i = k\nabla p_i$. *Provided that* $f \in W^{1,\infty}(L^p(\Omega)) \cap W^{2,p}(L^p(\Omega))$, $D_{tt}u(0) \in W^{1,p}(\Omega)$ *and* $D_{ttt}p(0) \in L^p(\Omega)$, *then the proof Lemma 2.6 in [51] implies that* $A_i(t,x) = \frac{\partial p}{\partial p_i} \in W^{1,\infty}(W^{1,p}(\Omega))$. *Consequently* $A_i(t,x) \in H^1(C^{1-\frac{N}{p}}(\Omega))$ *if* $p > N$ *by using Sobolev embedding theorem.*

As for the fully discrete scheme, we may use the following scheme (also see [41, 26]). The fully mixed formulation is to find $\{p_h^{n+1}, u_h^{n+1}\} \in Q_h \times \Sigma_h$ such that

$$\begin{cases} (D_{tt}p_h^n, w) - (divu_h^n, w) &= (f^n, w) \quad \forall w \in Q_h \\ (k^{-1}u_h^{n+1}, \chi) + (p_h^{n+1}, div\chi) &= 0 \quad \forall \chi \in \Sigma_h \\ (p_h^0, w) &= (g_0, w) \quad \forall w \in Q_h \\ (\frac{2}{\Delta t}D_t p_h^{\frac{1}{2}}, w) - (divu_h^0, w) &= (f^0 + \frac{2}{\Delta t}g_1, w) \quad \forall w \in Q_h \\ (u_h^0, \chi) &= (\Pi_h u(0), \chi) \quad \forall \chi \in RT_h^0, \end{cases} \tag{5.66}$$

where the difference notations are defined in section 5.3.3. It is known that the scheme in (5.66) is stable when $\Delta t \leq Ch$ and that the time consistence error $O(\Delta t^2)$ if $u(t,x)$ is smooth enough regarding to t (refer to [41, 26]). Consequently, we have by the global mixed MsFEM

$$\sup_{t_n} \|p - p_h^n\|_{L^2(\Omega)} + \sup_{t_n} \|u - u_h^n\|_{L_k^2(\Omega)} \leq C(h^{\min(1,\alpha+\beta_1-\beta_2-1)} + \Delta t^2).$$

5.5. Numerical Results

In this section, we present a few numerical results to demonstrate the importance of incorporating global information. We will consider multiscale finite element methods with limited global information presented in Section 5.2.

We will use a channelized permeability field $k(x)$ which induces strong non-local effects on a domain $\Omega = [0,1]^2$. This type of heterogeneities is presented in a benchmark test of the SPE comparative project [24] (upper Ness layers). These permeability fields are highly heterogeneous, channelized, and difficult to upscale. In Figure 4.1, we plot a log of this field. As can be observed, the irregular channels introduces strong non-locality across the entire domain. For these types of heterogeneities, local approaches usually fail to give an accurate results. In this experiment, we take $f = 10$, initial value $g_0 = 0$ and $g_1 = 0$. We impose zero Dirichlet boundary conditions in (5.2).

In our numerical tests, we compare the solutions at several time instances. Our first choice is standard MsFEM with local basis functions where linear boundary conditions are imposed for these basis functions. For the second approach, we used the MsFEM using limited global information, where where steady state solution is used to generate basis functions. Note that the construction of the basis functions is one time overhead that involves the solution of elliptic partial differential equation. Using these basis functions, the fine-scale equation can be solved repeatedly for various boundary conditions and right hand sides. For full discretization, the scheme in (5.41) is utilized and we use 220x60 fine grid and 22x6 coarse grid.

First, we present numerical results (Table 5.1 and 5.2) for relative errors for the solution u comparing (1) standard MsFEM (2) Oversampling MsFEM (3) MsFEM with limited global information at two different time instances. It is clear from these tables that MsFEM with limited global information performs much better than other local multiscale methods. This suggests that one needs to use some type of global information in constructing basis functions.

We depict the reference solution, the solution obtained local MsFEM and the improved MsFEM using limited global information in Figures 5.1 - 5.2 at two different

Table 5.1. Relative Errors at Time=0.4, Coarse Model = 22x10

Method	$L^2(\%)$	$H^1(\%)$	$L^\infty(\%)$
MsFEM	+2.03e+01	+4.94e+01	+3.52e+01
MsFEM-ovs	+6.36e+00	+2.51e+01	+1.44e+01
Global MsFEM	+4.33e+00	+1.98e+01	+9.92e+00

Table 5.2. Relative Errors at Time=0.6, Coarse Model = 55x15

Method	$L^2(\%)$	$H^1(\%)$	$L^\infty(\%)$
MsFEM	+1.95e+01	+3.53e+01	+3.03e+01
MsFEM-ovs	+3.56e+00	+1.50e+01	+8.48e+00
Global MsFEM	+2.46e+00	+1.08e+01	+4.78e+00

times $t = 0.4$ and $t = 0.6$. It is clear from these figures that MsFEM with limited global information provides accurate solution for such heterogeneous fields, where the localized methods fail. In this section, we restricted ourselves to only a few numerical results.

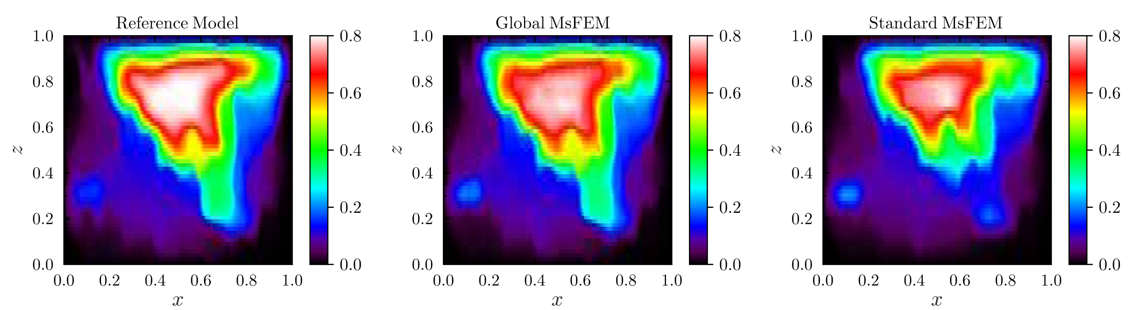

Fig. 5.1. Solution u_h at $t = 0.4$. Left: Reference solution. Middle: solution obtained by global MsFEM. Right: solution by local MsFEM

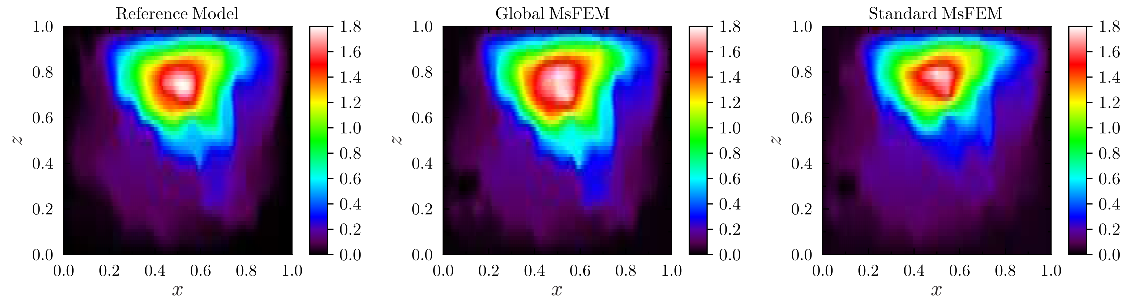

Fig. 5.2. Solution u_h at $t = 0.6$. Left: Reference solution. Middle: solution obtained by global MsFEM. Right: solution by local MsFEM

5.6. Conclusions and Comments

In this chapter, we study multiscale approaches for solving acoustic wave equations with heterogeneous coefficients without scale separation. The solution is approximated on a coarse grid using multiscale basis functions. For the construction of these basis functions, we employ global functions. In particular, these global fields are defined such that the solution smoothly depends on these fields. We provide analysis of the proposed methods for Galerkin finite element and mixed finite element formulations, and present a few numerical examples. The numerical results demonstrate that the solution can be captured accurately on a coarse grid when some type of limited global information is used.

CHAPTER VI

CONCLUSIONS

6.1. Conclusions

In this dissertation, we present and focus on the following multiscale numerical methods using limited global information: MsFEM, mixed MsFEM and multiscale PUM. The global information is represented by a set of functions. These functions can be thought as auxiliary functions which contain essential information about the heterogeneities. The computations of the global information are performed off-line. The finite element basis functions constructed employing the global information can be used to solve flow equations in heterogeneous porous media with different source terms, boundary conditions or mobility (see Section 2.1) on the coarse grid. The computation of global fields requires solving single-phase equations, thus the proposed approaches are effective when flow equations are solved repeatedly. One can use local solutions instead of global solutions in the proposed formulation of these global multiscale numerical methods. In this respect, the proposed global multiscale numerical methods will be similar to the local multiscale methods introduced in [38, 22] where the resonance error will appear in the convergence rates.

In this dissertation, we apply the proposed global multiscale numerical methods to elliptic, parabolic and wave equations with continuum spatial scales. We present rigorous analysis of the proposed multiscale numerical methods for the partial differential equations. We show that these methods are stable and converge without the resonance error. Assumption that the solution of the partial differential equation smoothly depends on the global fields, we construct multiscale finite element basis functions which contain these global information (fields). We discuss several cases where the global fields can be found and it can be shown that the solution smoothly depends on these global fields. We present a few preliminary numerical results to demonstrate the accuracy of the proposed approaches. We also consider a parameter dependent permeability field where a limited number of parameter value is used to generate the global fields. Based on these global fields, we compute multiscale basis functions. Basis functions are computed using single-phase flow solutions and these basis functions are used for the simulations of two-phase flow and transport. We consider SPE Comparative Project [24]. These permeability fields are channelized and hard to upscale and, therefore, single-phase flow information (limited global information) is used. Our numerical results show that one can achieve much better accuracy with a few global fields corresponding to single-phase flow solutions than those local multiscale finite element methods.

6.2. Future Work

While we have presented several multiscale numerical methods using limited global information for partial differential equations, we believe there are still many interesting areas for further research.

One of the areas for further study is to investigate the global multiscale numerical methods on unstructured grids. From the analysis presented in Chapter III, IV and V, we may find the possibility that the global multiscale finite element methods can be extended to unstructured coarse grids. The use of unstructured coarse grids has advantages in subsurface simulations since they provide flexibility and can render more accurate upscaled solutions for flow and transport equations. It is often necessary to use an unstructured coarse grid when highly heterogeneous reservoirs are discretized via irregular anisotropic fine grids. Although we have discussed global mixed MsFEM on unstructured grids and presented some encouraging results in [3], there is room for further exploration of some of the underlying approaches. As our intent in [3] was to demonstrate that mixed MsFEM on an unstructured grid could be effectively used in a full coarse-scale model, we did not consider the issue related to the upscaling of the saturation equation. As we noticed that the main source of errors in our coarse-scale simulations is due to the saturation upscaling. The upscaling of the saturation is achieved via the use of a carefully selected coarse grid, however, no subgrid model is used in the upscaling of the transport equations. We plan to investigate the subgrid effects in the saturation equation in the future. This will help to improve further the upscaling result. It may also be worth exploring the impact of the unstructured grid obtained via the single-phase flow on the accuracy of the mixed MsFEM. As we noticed that the mixed MsFEM is more accurate when an unstructured coarse grid is used. Further theoretical investigation of these issues is worth pursuing in future. In the meanwhile, it would be interesting in the future to study the global multiscale PUM and other global multiscale numerical methods on unstructured grids.

It would be also interesting in the future to consider nonlinear partial differential equations. We have considered the global multiscale numerical methods only for linear partial differential equations and have not discussed nonlinear partial differential equations in the dissertation. In the formulations of the global multiscale methods, we assume the solutions of partial differential equations smoothly depend on the predefined global fields (information). However, we have not yet found such global fields for nonlinear partial differential equations (e.g. nonlinear elliptic equations). We believe this would be a challenging problem in the future. Once we find the global fields, we can use the frameworks of the global multiscale methods to numerically solve the nonlinear partial differential equations and remove resonance error.

Another future direction is to consider domain decomposition methods for multiscale partial differential equations using limited global information. Domain decomposition refers to the splitting of a partial differential equation into coupled problems on smaller subdomains forming a partition of the original domain. We believe using global information would yield a robust coarsening when domain decomposition methods are applied to solve multiscale partial differential equations with continuum scales. It would be interesting in the future to theoretically study domain decomposition methods using limited global information and test the performance of the methods in numerical experiments.

REFERENCES

[1] J. Aarnes, On the use of a mixed multiscale finite element method for greater flexibility and increased speed or improved accuracy in reservoir simulation, SIAM MMS 2 (3) (2004) 421–439.

[2] J. Aarnes and Y. Efendiev, Mixed multiscale finite element for stochastic porous media flows, SIAM Sci. Comp. (2008) submitted.

[3] J. Aarnes, Y. Efendiev, T. Y. Hou, and L. Jiang, Mixed multiscale finite element methods on adaptive unstructured grids using limited global information: a chapter in Bridging the Scales in Science and Engineering, Oxford University Press, 2008 submitted.

[4] J. Aarnes, Y. Efendiev, and L. Jiang, Some multiscale results using limited global information for two-phase flow simulations, International Journal of Numerical Analysis and Modeling (2008) submitted.

[5] J. Aarnes, Y. Efendiev, and L. Jiang, Mixed multiscale finite element methods using limited global information, SIAM, MMS 7 (2) (2008) 655–676.

[6] R. A. Adams, Sobolev spaces, Academic Press, New York-London, 1975.

[7] G. Alessandrini and V. Nesi, Univalent σ-harmonic mappings, Archive for Rational Mechanics and Analysis 158 (2001) 155–171.

[8] G. Allaire and R. Brizzi, A multiscale finite element method for numerical homogenization, SIAM MMS 4 (2005) 790–812.

[9] T. Arbogast, Implementation of a locally conservative numerical subgrid upscaling scheme for two-phase Darcy flow, Comput. Geosci. 6 (3-4) (2002) 453–481.

[10] T. Arbogast and K. Boyd, Subgrid upscaling and mixed multiscale finite elements, SIAM Journal on Numerical Analysis 44 (3) (2006) 1150–1171.

[11] T. Arbogast, G. Pencheva, M. F. Wheeler, and I. Yotov, A multiscale mortar mixed finite element method, SIAM MMS 6 (2007) 319–346.

[12] I. Babuška, U. Banerjee, and J. E. Osborn, Survey of meshless and generalized finite element methods: a unified approach, Acta Numerica 12 (2003) 1–125.

[13] I. Babuška, U. Banerjee, and J. E. Osborn, Generalized finite element methods: main ideas, results and perspective, International Journal of Computational Methods 1 (2004) 67–103.

[14] I. Babuška, G. Caloz, and J. E. Osborn, Special finite element methods for a class of second order elliptic problems with rough coefficients , SIAM J. Numer. Anal. 31 (1994) 945–981.

[15] I. Babuška and J.E. Osborn, Generalized finite element methods: Their performance and their relation to mixed methods, SIAM J. Numer. Anal. 20 (1983) 510–536.

[16] I. Babuška and J. M. Melenk, The partition of unity method, Internat. J. Numer. Methods Engrg. 40 (4) (1997) 727–758.

[17] S. C. Brenner and L. R. Scott, The mathematical theory of finite element methods: texts in Applied Mathematics, vol. 15, Springer-Verlag, New York, 2002.

[18] F. Brezzi, Interacting with the subgrid world: numerical analysis 1999 (Dundee), Chapman & Hall/CRC, Boca Raton, FL, 2000, 69–82.

[19] F. Brezzi and M. Fortin, Mixed and hybrid finite element methods, Springer-Verlag, Berlin-Heidelberg-New York, 1991.

[20] Y. Chen and L. J. Durlofsky, An ensemble level upscaling approach for efficient estimation of fine scale production statistics using coarse scale simulations, SPE paper 106086 presented in SPE Reservoir Simulation Sympoisum, Houston, February (2007).

[21] Y. Chen, L. J. Durlofsky, M. Gerritsen, and X. H. Wen, A coupled local-global upscaling approach for simulating flow in highly heterogeneous formations, Advances in Water Resources 26 (2003) 1041–1060.

[22] Z. Chen and T. Y. Hou, A mixed multiscale finite element method for elliptic problems with oscillating coefficients, Math. Comp. 72 (242) (2002) 541–576 (electronic).

[23] Z. Chen, G. Huan, and Y. Ma, Computational methods for multiphase flows in porous media, Society for Industrial and Applied Mathematics, Philadelphia, 2006.

[24] M. Christie and M. Blunt, Tenth SPE Comparative Solution Project: a comparison of upscaling techniques, SPE Reser. Eval. Eng. 4 (2001) 308–317.

[25] P. G. Ciarlet, The finite element method for elliptic problems, Horth-Holland, Amsterdam, 1978.

[26] L. Cowsar, T. Dupont, and M. Wheeler, A priori estimates for mixed finite element methods for the wave equation, Comput. Methods Appl. Mech. Engrg. 82 (1990) 205–222.

[27] P. Dostert, Y. Efendiev, and T. Y. Hou, Multiscale finite element methods for stochastic porous media flow equations, CMMAME (2008) submitted.

[28] T. Dupont, L^2 estimates for Galerkin methods for second-order hyperbolic problems, SIAM J. Numer. Anal. 10 (1973) 880–889.

[29] Y. Efendiev, V. Ginting, T. Y. Hou, and R. Ewing, Accurate multiscale finite element methods for two-phase flow simulations, Journal of Computational Physics 220 (1) (2006) 155–174.

[30] Y. Efendiev, T. Hou, and V. Ginting, Multiscale finite element methods for nonlinear problems and their applications, Comm. Math. Sci. 2 (4) (2004) 553–589.

[31] Y. Efendiev, T. Y. Hou, and W. Luo, Preconditioning Markov Chain Monte Carlo simulations using coarse-scale models, SIAM SCI 28 (2) (2006) 776–803.

[32] Y. Efendiev, T. Y. Hou, and X. H. Wu, Convergence of a nonconforming multiscale finite element method, SIAM Journal on Numerical Analysis 37 (3) (2000) 888–910.

[33] Y. Efendiev and A. Pankov, Numerical homogenization of nonlinear random parabolic operators, SIAM Multiscale Modeling and Simulation 2(2) (2004) 237–268.

[34] A. Ern and J. Guermond, Theory and practice of finite elements, Springer-Verlag, New York, 2004.

[35] L. Evans, Partial differential equations, graduate studies in mathematics 19, American Mathematical Society, Providence, RI, 1998

[36] G. Fairweather, Finite element Galerkin methods for differential equations, Marcel Dekker, Inc., New York-Basel, 1978.

[37] K. Höllig, Finite element methods with B-splines, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2003.

[38] T. Y. Hou and X. H. Wu, A multiscale finite element method for elliptic problems in composite materials and porous media, Journal of Computational Physics 134 (1997) 169–189.

[39] T. Y. Hou, X. H. Wu, and Z. Cai, Convergence of a multiscale finite element method for elliptic problems with rapidly oscillating coefficients, Math. Comp. 68 (1999) 913–943.

[40] T. Hughes, G. Feijoo, L. Mazzei, and J. Quincy, The variational multiscale method - a paradigm for computational mechanics, Comput. Methods Appl. Mech. Engrg 166 (1998) 3–24.

[41] E. Jenkins, B. Riviere, and M. Wheeler, A priori error estimates for mixed finite element approximations of the acoustic wave equation, SIAM J. Numer. Anal. 40 (2002) 1698–1715.

[42] P. Jenny, S. H. Lee, and H. Tchelepi, Adaptive multi-scale finite volume method for multi-phase flow and transport in porous media, Multiscale Modeling and Simulation 3 (2005) 30–64.

[43] P. Jenny, S.H. Lee, and H. Tchelepi, Multi-scale finite volume method for elliptic problems in subsurface flow simulation, J. Comput. Phys. 187 (2003), 47–67.

[44] L. Jiang, Y. Efendiev, and V. Ginting, Global multiscale methods for acoustic wave equations with continum scales, SIAM, J. Numerical Analysis (2008) submitted.

[45] L. Jiang, Y. Efendiev, and V. Ginting, Multiscale methods for parabolic equations with continuum spatial scales, DCDS-B 8 (4) (2007) 833–859.

[46] V. V. Jikov, S. M. Kozlov, and O. A. Oleinik, Homogenization of differential operators and integral functionals, Springer-Verlag, New York, 1994.

[47] Y. Kuznetsov and S. Repin, Convergence analysis and error estimates for mixed finite element method on distorted meshes, J. Numer. Math. 13 (1) (2005) 33–51.

[48] A. M. Matache and C. Schwab, Homogenization via p-FEM for problems with microstructure, Proceedings of the Fourth International Conference on Spectral and High Order Methods (ICOSAHOM 1998) (Herzliya), vol. 33, 2000, pp. 43–59.

[49] J. M. Melenk and I. Babuška, The partition of unity finite element method: basic theory and applications, Comput. Methods Appl. Mech. Engrg. 139 (1996) 289–314.

[50] S. Moskow and M. Vogelius, First order corrections to the homogenized eigenvalues of a periodic composite medium, a convergence proof, Proc. Roy. Soc. Edinburgh 127 (1997) 1263-1299.

[51] H. Owhadi and L. Zhang, Homogenization of the acoustic wave equation with a continuum of scales, arXiv, math.AP/0604380 (2007).

[52] H. Owhadi and L. Zhang, Homogenization of parabolic equations with a continuum of space and time scales, SIAM Journal on Numerical Analysis 46 (1)

(2007) 1–36.

[53] H. Owhadi and L. Zhang, Metric based up-scaling, Comm. Pure and Applied Math. 60 (2007) 675–723.

[54] B. G. Pachpatte, Mathematical inequalities, Elsevier, Amsterdam-Tokyo, 1952.

[55] A. Quarteroni and A. Valli, Numerical approximation of partial differential equations, Springer-Verlag, Berlin-New York, 1994.

[56] G. Sangalli, Capturing small scales in elliptic problems using a residual-free bubbles finite element method, Multiscale Model. Simul. 1 (3) (2003) 485–503 (electronic).

[57] J. Shen, Mixed finite element methods on distorted rectangular grids, Technical Report Series in ISC, TAMU ISC-94-13 (1994).

[58] T. Strouboulis, L. Zhang, and I. Babuška, Generalized finite element method using mesh-based handbooks: application to problems in domains with many voids. multiscale computational mechanics for materials and structures, Comput. Methods Appl. Mech. Engrg. 192 (2003) 3109–3161.

[59] X.H. Wu, R. Parashkevov, and M. Stone, Reservoir modeling with global scale-up, SPE 105237, presented in the 15th Middle East Oil & Gas Show and Conference, March 11-14, 2007.

VITA

Lijian Jiang was born in Xiangtan, Hunan Province, P. R. China. He received his Bachelor of Science in Mathematics from Hunan Normal University in June 1998. He then began studies at Institute of Mathematics, Chinese Academy of Sciences where he received his Master of Science in Mathematics in June 2001. He started studies at Texas A&M University in September 2002 and received his Ph.D. from the Department of Mathematics in August 2008. Lijian Jiang can be contacted by writing to: Institute for Mathematics and its Applications, University of Minnesota, 114 Lind Hall, 207 Church Street S.E., Minneapolis, MN 55455-0134 or by the email address ljjiang16@hotmail.com.